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Timelike entanglement entropy and $T\bar{T}$ deformation

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Abstract

In a previous work [1] about the $T\bar{T}$ deformed CFT$_2$, from the consistency requirement of the entanglement entropy, we found that in addition to the usual spacelike entanglement entropy, a timelike entanglement entropy must be introduced and treated equally. Motivated by the recent explicit constructions of the timelike entanglement entropy and its bulk dual, we provide a comprehensive analysis of the timelike and spacelike entanglement entropies in the $T\bar{T}$ deformed finite size system and finite temperature system. The results confirm our prediction that in the finite size system only the timelike entanglement entropy receives a correction, while in the finite temperature system only the usual spacelike entanglement entropy gets a correction. These findings affirm the necessity of a complete measure including both spacelike and timelike entanglement entropies, referred to as the general entanglement entropy, for characterizing deformed systems from the quantum information perspective.

Contents

1 Introduction

Conformal field theories (CFTs) could be deformed by relevant, marginal and irrelevant deformations. Relevant and marginal deformations have been well studied. Though the irrelevant deformations are non-renormalizable and lead no consequence in the IR region, in two-dimensional spacetime, they turn out to be generally under control and even solvable for some particular models. One such solvable irrelevant deformation is the $T\bar{T}$ deformation [2–4], obtained by turning on a $T\bar{T}$ coupling term

$$\frac{dI^{(\mu)}_{\mathrm{QFT}}}{d\mu} = \int d^2x \, \left(T\bar{T}\right)_{\mu} , \tag{1.1}$$

where the deformation parameter μ has the $[\mathrm{Length}]^2$ dimension and $\left(T\bar{T}\right)_{\mu}$ is defined by the stress tensor of the deformed theory. At the leading order, the deformed theory is given by

$$I^{(\mu)}_{\mathrm{QFT}} = I_{\mathrm{CFT}} + \mu \int d^2x \, \left(T\bar{T}\right)_{\mu=0} + \mathcal{O}(\mu^2), \tag{1.2}$$

where $(T\bar{T})_{\mu=0} = \frac{1}{8}\left[T^{\alpha\beta}T_{\alpha\beta} - (T^{\alpha}_{\alpha})^2\right]_{\mu}$. It is clear that the Lorentz symmetry is preserved but the conformal symmetry is broken in the $T\bar{T}$ deformed theory.

The importance of this model is revived partially by the proposition, given in Ref. [5], that the AdS_3 gravity with a Dirichlet boundary at a finite radial distance r_c, is dual to the $T\bar{T}$ deformed CFT_2 living on that Dirichlet boundary. This nontrivial extension of AdS/CFT correspondence [6] is called cutoff-AdS/$T\bar{T}$-deformed-CFT (cAdS/dCFT) correspondence.

In the UV region, out of the many calculable deformed physical quantities, a particularly important one is the *entanglement entropy* (EE). Until recently, the entanglement entropy is defined only for spacelike intervals.

Thereinafter, we will refer the spacelike EE to the usual standard EE. Some progresses on the spacelike EE in the $T\bar{T}$ deformed CFT_2 have been achieved in recent years [7–9].

With the replica trick [10,11], in Ref. [12], the $T\bar{T}$ deformed spacelike EE was calculated perturbatively for the cylindrical topology. Intriguingly, the $T\bar{T}$ correction to the spacelike EE is dependent on different interpretations of the identical topology. When treat the system as a finite size one, there is no leading correction. But the finite temperature interpretation does receive a leading correction. This indicates that the $T\bar{T}$ correction to the spacelike EE can be observed in the finite temperature system but is invisible in the finite size system. This result obviously conflicts with the fact that the entanglement entropy is a topological quantity. Moreover, without a correction presented in the finite size system, how do we distinguish between the undeformed CFT and the deformed CFT by the entanglement entropy?

To resolve this inconsistency, in a previous paper [1], by noticing the fact that the finite size system and the finite temperature system share the same cylindrical topology under exchanging $t \leftrightarrow x$, we proposed that in addition to the spacelike EE, there should exist a timelike EE for timelike intervals, and the timelike EE should be treated on the same footing as the spacelike EE. We further predicted, as shown in Table 1, the spacelike EE only receives a correction in the $T\bar{T}$ deformed finite temperature system, while the timelike EE only receives a correction in the $T\bar{T}$ deformed finite size system.

	spacelike EE	timelike EE
Finite size		✓
Finite temperature	✓	

Table 1: The symbol ✓ marks a correction to the entanglement entropy caused by the $T\bar{T}$ deformation.

Remarkably, such a timelike EE has been specifically defined via analytical continuation and the bulk dual has been explicitly provided in Ref. [13–16] recently. Rather than real-valued as the spacelike EE, the timelike EE is a complex-valued quantity. It is suggested that the timelike EE needs to be correctly understood as a pseudoentropy, which is a non-Hermitian generalization of the usual spacelike EE. According to the Ryu-Takayanagi formula [17], the bulk dual of the pseudoentropy is represented by a complex-valued extremal surface, which was originally proposed in [18–20]. For a $2d$ CFT that is dual to the Poincare patch of AdS_3, the timelike EE S_A of a timelike interval A, whose width is given by T, reads

$$S_A = \frac{c_{\mathrm{AdS}}}{3} \log\left(\frac{T}{\epsilon}\right) + i\frac{c_{\mathrm{AdS}}\pi}{6}, \tag{1.3}$$

with ϵ the UV cutoff and $c_{\mathrm{AdS}} = 3R_{\mathrm{AdS}}/2G_N$ the central charge of CFT_2. Ref. [13,21] found that the complex-valued extremal surface consists of one timelike geodesic and two spacelike geodesics, as shown in Figure 1. Two spacelike geodesics connect ∂A and null infinities, respectively. The timelike geodesic connects the endpoints of two spacelike geodesics on null infinities. Furthermore, the length of the timelike geodesic is equal to the imaginary part of the timelike EE, while the total length of two spacelike geodesics is equal to the real part of the timelike EE.

The purpose of this paper is to confirm the predictions made in [1]. To this end, we will construct a *general entanglement entropy*, which contains both spacelike and timelike components, to describe the $T\bar{T}$ deformation from a quantum information perspective. In the cAdS/dCFT correspondence, the general entanglement entropy

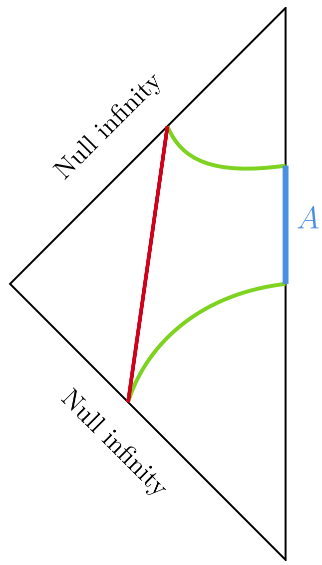

Figure 1: Geodesics connecting to ∂A in the Poincare patch of AdS_3. The red line denotes one timelike geodesic and two green lines denote two spacelike geodesics. The blue line denotes a timelike interval A on the conformal boundary.

could be thought of a complete measure, which always receives a correction from the $T\bar{T}$ deformation. The spacelike and timelike entanglement entropies are different limits of the general entanglement entropy. As a consistent check, we will show that, the leading $T\bar{T}$ correction to the timelike EE exists in the finite size system, but vanishes in the finite temperature system. Utilizing the holographic method, we will show the physical reason why spacelike(timelike) EE only receives a correction in finite temperature(size) system.

The remainder of this paper is outlined as follows. In section 2, we show the leading correction of the general entanglement entropy of the $T\bar{T}$ deformed finite temperature CFT_2 and finite size CFT_2, respectively. In section 3, using RT formula, we show that the cutoff-AdS geodesic leads to a precise estimation of the general entanglement entropy in $T\bar{T}$ deformed CFT. Section 4 is for conclusion.

2 General entanglement entropy in $T\bar{T}$ deformed CFT

In this section, we calculate the leading corrections to the general entanglement entropy of the $T\bar{T}$ deformed CFT living on the cylindrical manifold. After taking different limits, we derive the corrections to the spacelike EE and timelike EE respectively, for the finite temperature system and finite size system. For a $T\bar{T}$ deformed CFT on $\mathcal{M}$, the entanglement entropy of some subsystem $A \in \mathcal{M}$ is obtained by

$$S\left(A\right) = \lim_{n\to 1} S_n\left(A\right), \quad S_n\left(A\right) = \frac{1}{1-n}\log\frac{Z_n\left(A\right)}{Z^n}, \tag{2.4}$$

with Z the partition function on $\mathcal{M}$ and $Z_n\left(A\right)$ the partition function on $\mathcal{M}^n$ obtained by gluing n copies of $\mathcal{M}$ along A. Substituting equation (1.2) into equation (2.4), one could obtain the leading correction to $S\left(A\right)$:

$$\delta S\left(A\right) = \lim_{n\to 1}\delta S_n\left(A\right), \quad \delta S_n\left(A\right) = \frac{n\mu}{1-n}\int_{\mathcal{M}}\left[\left\langle T\bar{T}\right\rangle_{\mathcal{M}} - \left\langle T\bar{T}\right\rangle_{\mathcal{M}^n}\right]. \tag{2.5}$$

2.1 Finite temperature

Consider a $T\bar{T}$ deformed CFT at the inverse temperature β. This theory lives on a cylindrical manifold $\mathcal{M}_1$, which has a non-compact spatial direction $x \in (-\infty, \infty)$ and compact Euclidean time $\tau \in (0, \beta)$ with the periodicity $\tau \sim \tau + \beta$. It is well-known that the two-point correlation function in the finite temperature CFT is

$$\langle \mathcal{O}(w, \bar{w}) \mathcal{O}(0,0) \rangle = \left[\frac{\beta^2}{\pi^2} \sinh\left(\frac{\pi w}{\beta} \right) \sinh\left(\frac{\pi \bar{w}}{\beta} \right) \right]^{-2\Delta} \tag{2.6}$$

with the complex coordinate $w = x + i\tau$ and the scaling dimension Δ. By the replica trick, the entanglement entropy of a single interval A, which has timelike width τ_0 and spacelike width x_0, is related to the two-point function of the twist fields

$$\begin{aligned} \frac{Z_n(A)}{Z^n} &= \left\langle \Phi^+(0,0) \Phi^-(x_0 + i\tau_0, x_0 - i\tau_0) \right\rangle \\ &= \left[\frac{\beta^2}{\pi^2 \epsilon^2} \sinh\left(\frac{\pi(x_0 + i\tau_0)}{\beta} \right) \sinh\left(\frac{\pi(x_0 - i\tau_0)}{\beta} \right) \right]^{-2\Delta_n} \end{aligned}$$

with the dimension$\Delta_n = \frac{c_{\text{AdS}}}{24}\left(n - \frac{1}{n}\right)$, the UV cutoff ϵ and the central charge c_{AdS}. Therefore, the entanglement entropy obtained by equation (2.4) is

$$\begin{aligned} S(A) &= \frac{c_{\text{AdS}}}{6} \log \left[\frac{\beta^2}{\pi^2 \epsilon^2} \sinh\left(\frac{\pi(x_0 + i\tau_0)}{\beta} \right) \sinh\left(\frac{\pi(x_0 - i\tau_0)}{\beta} \right) \right] \\ &= \frac{c_{\text{AdS}}}{3} \log \left[\frac{\beta}{2\pi\epsilon} \sqrt{2\cosh\left(\frac{2\pi x_0}{\beta} \right) - 2\cosh\left(\frac{2\pi\tau_0}{\beta} \right)} \right]. \end{aligned} \tag{2.7}$$

By the replica trick, one could also calculate $\langle T\bar{T} \rangle_{\mathcal{M}_1}$ and $\langle T\bar{T} \rangle_{\mathcal{M}_1^n}$ in the finite temperature CFT:

$$\langle T\bar{T} \rangle_{\mathcal{M}_1} = \left(\frac{c_{\text{AdS}}}{12} \right)^2 \left(\frac{2\pi^2}{\beta^2} \right)^2, \tag{2.8}$$

$$\langle T\bar{T} \rangle_{\mathcal{M}_1^n} = \left(\frac{c_{\text{AdS}}}{12} \right)^2 \left[\left(\frac{2\pi^2}{\beta^2} \right)^2 - \frac{2\pi^2}{\beta^2}(n-1)\left(Q + \bar{Q}\right) + O\left((n-1)^2 \right) \right], \tag{2.9}$$

with the meromorphic function

$$Q(w) := \frac{\sinh^2\left(\frac{\pi(x_0 + i\tau_0)}{\beta} \right)}{\sinh^2\left(\frac{\pi(x_0 + i\tau_0 - w)}{\beta} \right) \sinh^2\left(\frac{\pi w}{\beta} \right)}.$$

Intuitively, $Q(w)$ has two poles ($w = 0$ and $w = x_0 + i\tau_0$) that correspond to the residues

$$\begin{aligned} \text{Res}(Q, w = 0) &= \frac{2\pi}{\beta} \coth\left(\frac{\pi(x_0 + i\tau_0)}{\beta} \right), \\ \text{Res}(Q, w = x_0 + i\tau_0) &= -\frac{2\pi}{\beta} \coth\left(\frac{\pi(x_0 + i\tau_0)}{\beta} \right). \end{aligned}$$

The leading correction to $S(A)$ caused by $T\bar{T}$ deformation is thus

$$\delta S(A) = \lim_{n\to1} \frac{n\mu}{1-n} \int_{\mathcal{M}_1} \left(\frac{c_{\text{AdS}}}{12} \right)^2 \left[\frac{2\pi^2}{\beta^2}(n-1)\left(Q + \bar{Q}\right) + O\left((n-1)^2 \right) \right] \tag{2.10}$$

$$= -\mu \int_{\mathcal{M}_1} \left(\frac{c_{\text{AdS}}}{12} \right)^2 \frac{2\pi^2}{\beta^2} \left(Q + \bar{Q}\right), \tag{2.11}$$

which, with the help of Cauchy's residue theorem, could be simplified as

$$\begin{aligned}\delta S(A) &= -2\pi\mu\int_0^{x_0} dx\left(\frac{c_{\text{AdS}}}{12}\right)^2\frac{2\pi^2}{\beta^2}\times\left[\text{Res}\left(Q,0\right)+\text{Res}\left(\bar{Q},0\right)\right],\\ &= -\mu\frac{\pi^4 c_{\text{AdS}}^2}{18\beta^3}x_0\left[\coth\left(\frac{\pi\left(x_0+i\tau_0\right)}{\beta}\right)+\coth\left(\frac{\pi\left(x_0-i\tau_0\right)}{\beta}\right)\right]\\ &= -\mu\frac{\pi^4 c_{\text{AdS}}^2}{9\beta^3}\frac{x_0\sinh\left(\frac{2\pi x_0}{\beta}\right)}{\cosh\left(\frac{2\pi x_0}{\beta}\right)-\cosh\left(\frac{2\pi i\tau_0}{\beta}\right)}.\end{aligned}$$

Via an analytical continuation $\tau = it$, the entanglement entropy in the finite temperature system, to the leading order of μ, reads

$$S^{(\mu)}\left(A\right)=\frac{c_{\text{AdS}}}{3}\log\left[\frac{\beta}{2\pi\epsilon}\sqrt{2\cosh\left(\frac{2\pi x_0}{\beta}\right)-2\cosh\left(\frac{2\pi t_0}{\beta}\right)}\right]-\mu\frac{\pi^4 c_{\text{AdS}}^2}{9\beta^3}\frac{x_0\sinh\left(\frac{2\pi x_0}{\beta}\right)}{\cosh\left(\frac{2\pi x_0}{\beta}\right)-\cosh\left(\frac{2\pi t_0}{\beta}\right)}. \tag{2.12}$$

By choosing $(x_0, t_0) = (0, T)$, one obtains the timelike entanglement entropy

$$S_T^{(\mu)}=\frac{c_{\text{AdS}}}{3}\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi T}{\beta}\right)\right]+i\frac{c_{\text{AdS}}\pi}{6}. \tag{2.13}$$

It is not surprising that, in the $T\bar{T}$ deformed finite temperature CFT$_2$, the timelike entanglement entropy does not receive a correction from the $T\bar{T}$ deformation. Moreover, it is worthwhile to note that the imaginary part of the timelike entanglement entropy originates from the complex logarithmic function. Despite the multivalued property, one can only take the principal value to match the total length of extremal geodesics in the bulk [21]. Similarly, choosing $(x_0, t_0) = (X, 0)$, the spacelike entanglement entropy with $T\bar{T}$ correction is determined:

$$S_X^{(\mu)}\left(A\right)=\frac{c_{\text{AdS}}}{3}\log\left[\frac{\beta}{\pi\epsilon}\sinh\left(\frac{\pi X}{\beta}\right)\right]-\mu\frac{\pi^4 c_{\text{AdS}}^2}{9\beta^3}X\coth\left(\frac{\pi X}{\beta}\right), \tag{2.14}$$

which exactly agrees with the result in [12].

2.2 Finite size

Now we focus on the $T\bar{T}$ living on a cylindrical manifold $\mathcal{M}_2$, which has a non-compact temporal direction $\tau \in (-\infty, \infty)$ and compact spatial direction $x \in (0, L)$ with the periodicity $x \sim x + L$. It is important to notice that $\mathcal{M}_1$ and $\mathcal{M}_2$ have the same topology $R \times S^1$. Therefore, by setting $\beta = L$ and exchanging $x \leftrightarrow \tau$ in equations (2.7) and (2.10), after performing the analytical continuation $\tau = it$, one easily obtains the entanglement entropy in the finite size system

$$S^{(\mu)}\left(A\right)=\frac{c_{\text{AdS}}}{3}\log\left[\frac{L}{2\pi\epsilon}\sqrt{2\cos\frac{2\pi t_0}{L}-2\cos\frac{2\pi x_0}{L}}\right]+\mu\frac{\pi^4 c_{\text{AdS}}^2}{9L^3}\frac{t_0\sin\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right)-\cos\left(\frac{2\pi x_0}{L}\right)}. \tag{2.15}$$

Setting $(x_0, t_0) = (0, T)$, the timelike entanglement entropy indeed receives a correction from the $T\bar{T}$ deformation

$$S_T^{(\mu)}\left(A\right)=\frac{c_{\text{AdS}}}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi T}{L}\right)\right]-\mu\frac{\pi^4 c_{\text{AdS}}^2}{9L^3}T\cot\left(\frac{\pi T}{L}\right)+i\frac{c_{\text{AdS}}\pi}{6}. \tag{2.16}$$

Meanwhile, as expected, the spacelike entanglement entropy does not receive a correction

$$S_X^{(\mu)}\left(A\right)=\frac{c_{\text{AdS}}}{3}\log\left[\frac{L}{\pi\epsilon}\sin\left(\frac{\pi X}{L}\right)\right]. \tag{2.17}$$

	EE of a single interval	Leading $T\bar{T}$ correction
Finite size	Timelike: $S_T = \frac{c_{\text{AdS}}}{3}\log\left(\frac{L}{\pi\epsilon}\sin\left(\frac{\pi T}{L}\right)\right) + \frac{ic_{\text{AdS}}\pi}{6}$	$\delta S_T = -\mu\frac{\pi^4 c_{\text{AdS}}^2}{9L^3}T\cot\left(\frac{\pi T}{L}\right)$
	Spacelike: $S_X = \frac{c_{\text{AdS}}}{3}\log\left(\frac{L}{\pi\epsilon}\sin\left(\frac{\pi X}{L}\right)\right)$	$\delta S_X = 0$
Finite temperature	Timelike: $S_T = \frac{c_{\text{AdS}}}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi T}{\epsilon}\right) + \frac{ic_{\text{AdS}}\pi}{6}$	$\delta S_T = 0$
	Spacelike: $S_X = \frac{c_{\text{AdS}}}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{\pi X}{\epsilon}\right)$	$\delta S_X = -\mu\frac{\pi^4 c_{\text{AdS}}^2}{9\beta^3}X\coth\left(\frac{\pi X}{\beta}\right)$

Table 2: The leading correction to the spacelike/timelike EE caused by the $T\bar{T}$ deformation in finite size/temperature system.

In the above field-theoretic results, the leading $T\bar{T}$ correction to the timelike EE exists in finite size systems but vanishes in finite temperature systems, while the leading $T\bar{T}$ correction to the spacelike EE exhibits the opposite behavior, as shown in Table 2, which is in perfect agreement with our prediction in [1]. Meanwhile, the leading $T\bar{T}$ correction to the general entanglement entropy always exists in both finite size systems and finite temperature systems. Therefore, the general entanglement entropy is the right measure to mark the deformations.

3 Gravity duals

It is illuminating to study the general entanglement entropy and its corresponding gravity dual in the context of cAdS/dCFT correspondence. In this section, we will demonstrate that the distance of the geodesic in the cutoff-AdS precisely matches the general entanglement entropy in the $T\bar{T}$ deformed CFT$_2$. Before that, we first

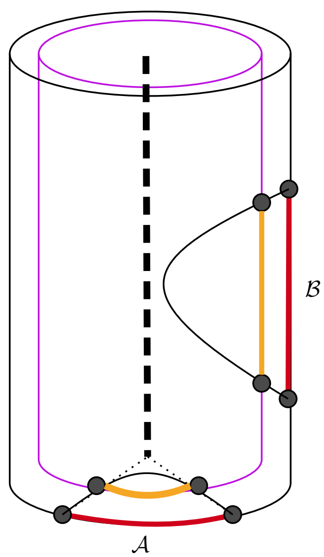

Figure 2: The figure illustrates a Euclidean cylindrical manifold with a black cylinder representing the original boundary and a violet cylinder representing the cut-off boundary. Two red lines denote an interval $\mathcal{A}$ in the compact direction and an interval $\mathcal{B}$ in the non-compact direction. Two orange lines indicate their corresponding intervals in the cut-off boundary.

provide a physical interpretation of our previous conjecture by introducing a Euclidean cylindrical manifold, as depicted in Figure 2, which features one compact and one non-compact direction. The compact direction is always described by a native parameter, either the inverse temperature β in the finite temperature system or the total length L in the finite size system. As a result, the interval $\mathcal{A}$ in the compact direction is independent of the cut-off boundary and the geodesic anchored on $\partial\mathcal{A}$ remains unchanged. The interval $\mathcal{A}$ in the compact direction is independent of the choice of the cut-off boundary, and the geodesic anchored on $\partial\mathcal{A}$ will not change. In contrast, the interval $\mathcal{B}$ in the non-compact direction will vary with the cut-off boundary, which leads to a change in the geodesic anchored on $\partial\mathcal{B}$. Therefore, in the finite temperature system, the temporal direction is compact and its timelike EE remains unaffected by the $T\bar{T}$ deformation; in the finite size system, the spatial direction is compact and its spacelike EE remains unchanged under the $T\bar{T}$ deformation.

3.1 BTZ black hole

Consider a BTZ black hole that is described by

$$ds^2 = -\frac{r^2 - r_+^2}{R_{\text{AdS}}^2}dt^2 + \frac{R_{\text{AdS}}^2}{r^2 - r_+^2}dr^2 + \frac{r^2}{R_{\text{AdS}}^2}dx^2, \tag{3.18}$$

where r_+ is the radius of event horizon, and t is the compact temporal direction $t \sim t + i\beta$. It is well-known that the finite temperature CFT_2 is dual to a BTZ black hole with the same temperature

$$\beta^{-1} = \frac{r_+}{2\pi R_{\text{AdS}}^2}, \tag{3.19}$$

In the cAdS/dCFT correspondence, the $T\bar{T}$ deformed CFT at finite temperature is dual to a BTZ black hole with the radial cutoff r_c,

$$r_c^2 = \frac{6R_{\text{AdS}}^4}{\pi c_{\text{AdS}}\mu}. \tag{3.20}$$

Taking $r = r_c$ and multiplying a factor $R_{\text{AdS}}^2/(r_c^2 - r_+^2)$ in the BTZ black hole metric, the metric of the cutoff boundary, where the $T\bar{T}$ deformed CFT lives, reads

$$ds^2 = -dt^2 + \frac{dx^2}{1 - r_+^2/r_c^2}. \tag{3.21}$$

Notice that the timelike interval on the boundary keeps invariant, because the compact temporal direction should be physical. This is the reason that the timelike EE does not receive a correction from the $T\bar{T}$ deformation. To see this, for an interval A, with the timelike width t_0 and the spacelike width x_0, in the $T\bar{T}$ deformed CFT, we calculate the general entanglement entropy with the RT formula. Performing the following coordinates transformation,

$$\begin{aligned} u &= \sqrt{1 - \frac{r_+^2}{r^2}} \cosh\left(\frac{r_+ t}{R_{\text{AdS}}^2}\right) \exp\left(\frac{r_+ x}{R_{\text{AdS}}^2}\right), \\ v &= \sqrt{1 - \frac{r_+^2}{r^2}} \sinh\left(\frac{r_+ t}{R_{\text{AdS}}^2}\right) \exp\left(\frac{r_+ x}{R_{\text{AdS}}^2}\right), \\ z &= \frac{r_+}{r} \exp\left(\frac{r_+ x}{R_{\text{AdS}}^2}\right), \end{aligned} \tag{3.22}$$

the metric (3.18) becomes

$$ds^2 = \frac{R_{\text{AdS}}^2}{z^2}\left(du^2 - dv^2 + dz^2\right). \tag{3.23}$$

Two boundary points $\partial A = \{(u_1, v_1, z_1), (u_2, v_2, z_2)\}$ could be written as

$$\begin{cases} u_1 = \sqrt{1-\frac{r_+^2}{r_c^2}}, & u_2 = \sqrt{1-\frac{r_+^2}{r_c^2}}\cosh\left(\frac{r_+ t_0}{R_{\rm AdS}^2}\right)\exp\left(\frac{r_+ x_0}{R_{\rm AdS}^2}\sqrt{1-\frac{r_+^2}{r_c^2}}\right), \\ v_1 = 0, & v_2 = \sqrt{1-\frac{r_+^2}{r_c^2}}\sinh\left(\frac{r_+ t_0}{R_{\rm AdS}^2}\right)\exp\left(\frac{r_+ x_0}{R_{\rm AdS}^2}\sqrt{1-\frac{r_+^2}{r_c^2}}\right), \\ z_1 = \frac{r_+}{r_c}, & z_2 = \frac{r_+}{r_c}\exp\left(\frac{r_+ x_0}{R_{\rm AdS}^2}\sqrt{1-r_+^2/r_c^2}\right). \end{cases} \tag{3.24}$$

Therefore, one can determine the length of the geodesic γ_A to be

$$\begin{aligned} L_{\gamma_A} &= R_{\rm AdS}\,\text{arcosh}\left[1+\frac{(u_2-u_1)^2-(v_2-v_1)^2+(z_2-z_1)^2}{2z_1z_2}\right] \\ &= R_{\rm AdS}\,\text{arcosh}\left[\frac{\left(r_+^2/r_c^2-1\right)\cosh\left(\frac{r_+t_0}{R_{\rm AdS}^2}\right)+\cosh\left(\frac{r_+x_0}{R_{\rm AdS}^2}\sqrt{1-r_+^2/r_c^2}\right)}{r_+^2/r_c^2}\right] \\ &= 2R_{\rm AdS}\log\left[\frac{r_c}{r_+}\sqrt{2\cosh\left(\frac{r_+x_0}{R_{\rm AdS}^2}\right)-2\cosh\left(\frac{r_+t_0}{R_{\rm AdS}^2}\right)}\right]-\frac{r_+^2}{r_c^2}\frac{\frac{r_+x_0}{2R_{\rm AdS}}\sinh(\frac{r_+x_0}{R_{\rm AdS}^2})}{\cosh(\frac{r_+x_0}{R_{\rm AdS}^2})-\cosh(\frac{r_+t_0}{R_{\rm AdS}^2})} \\ &\quad+\frac{r_+^2}{r_c^2}\frac{R_{\rm AdS}\,\cosh(\frac{r_+t_0}{R_{\rm AdS}^2})}{\cosh(\frac{r_+x_0}{R_{\rm AdS}^2})-\cosh(\frac{r_+t_0}{R_{\rm AdS}^2})}+O\left(\frac{r_+^3}{r_c^3}\right). \end{aligned} \tag{3.25}$$

Notice that the first term in the last line should not be thought of a leading correction caused by $T\bar{T}$ flow. To see this point, replacing the metric of the cutoff boundary with $ds^2 = -dt^2 + dx^2$, one can compute the length of the geodesic γ_A:

$$\begin{aligned} L_{\gamma_A} &= R_{\rm AdS}\,\text{arcosh}\left[\frac{\left(r_+^2/r_c^2-1\right)\cosh\left(\frac{r_+t_0}{R_{\rm AdS}^2}\right)+\cosh\left(\frac{r_+x_0}{R_{\rm AdS}^2}\right)}{r_+^2/r_c^2}\right] \\ &= 2R_{\rm AdS}\log\left[\frac{r_c}{r_+}\sqrt{2\cosh\left(\frac{r_+x_0}{R_{\rm AdS}^2}\right)-2\cosh\left(\frac{r_+t_0}{R_{\rm AdS}^2}\right)}\right] \\ &\quad+\frac{r_+^2}{r_c^2}\frac{R_{\rm AdS}\cosh(\frac{r_+t_0}{R_{\rm AdS}^2})}{\cosh(\frac{r_+x_0}{R_{\rm AdS}^2})-\cosh(\frac{r_+t_0}{R_{\rm AdS}^2})}+O\left(\frac{r_+^3}{r_c^3}\right). \end{aligned} \tag{3.26}$$

Therefore, utilizing equations (3.19) and (3.20) and identifying $\epsilon^2 = \pi c_{\rm AdS}\mu/6$, the right estimation of the general entanglement entropy corrected by $T\bar{T}$ deformation is

$$\frac{L_{\gamma_A}}{4G_N} \sim \frac{R_{\rm AdS}}{2G_N}\log\left[\frac{\beta}{2\pi\epsilon}\sqrt{2\cosh\left(\frac{2\pi x_0}{\beta}\right)-2\cosh\left(\frac{2\pi t_0}{\beta}\right)}\right]-\frac{R_{\rm AdS}}{2G_N}\frac{\pi^3 c_{\rm AdS}\mu}{3\beta^2}\frac{\frac{\pi x_0}{\beta}\sinh(\frac{2\pi x_0}{\beta})}{\cosh(\frac{2\pi x_0}{\beta})-\cosh(\frac{2\pi t_0}{\beta})}, \tag{3.27}$$

that is

$$S_A^{(\mu)} = \frac{c_{\rm AdS}}{3}\log\left[\frac{\beta}{2\pi\epsilon}\sqrt{2\cosh\left(\frac{2\pi x_0}{\beta}\right)-2\cosh\left(\frac{2\pi t_0}{\beta}\right)}\right]-\mu\frac{\pi^4 c_{\rm AdS}^2}{9\beta^3}\frac{x_0\sinh(\frac{2\pi x_0}{\beta})}{\cosh(\frac{2\pi x_0}{\beta})-\cosh(\frac{2\pi t_0}{\beta})} \tag{3.28}$$

This precisely matches the field theoretic result (2.12).

3.2 Global AdS_3

Consider the finite size CFT_2 at zero temperature that is dual to the global AdS_3. The metric of the global AdS_3 is given by

$$ds^2 = R_{\rm AdS}^2 \left(-\cosh^2\rho\, d\theta^2 + d\rho^2 + \sinh^2\rho\, d\phi^2\right), \tag{3.29}$$

where ϕ is the compact spatial direction $\phi \sim \phi + 2\pi$. In the cAdS/dCFT correspondence, the $T\bar{T}$ deformed CFT lives at the radial cutoff ρ_c,

$$\cosh^2\rho_c = \frac{3L^2}{2\mu\pi^3 c_{\rm AdS}}, \tag{3.30}$$

where the total length of the boundary circle is L, and the metric of the cutoff boundary reads

$$ds^2 = -\coth^2\rho_c\, d\theta^2 + d\phi^2. \tag{3.31}$$

Intriguingly, the timelike interval in the boundary is indeed changed by $T\bar{T}$ deformation, which means that timelike EE in the finite size system will be corrected by $T\bar{T}$ deformation. By choosing an interval A with the timelike width $t_0 = \frac{L\theta_0}{2\pi}$ and the spacelike width $x_0 = \frac{L\phi_0}{2\pi}$, and performing the following coordinates transformation,

$$\begin{aligned} u &= \tanh\rho\cos(\theta)\exp(i\phi), \\ v &= \tanh\rho\sin(\theta)\exp(i\phi), \\ z &= \frac{1}{\cosh\rho}\exp(i\phi), \end{aligned} \tag{3.32}$$

the metric (3.29) becomes

$$ds^2 = \frac{R_{\rm AdS}^2}{z^2}\left(du^2 + dv^2 + dz^2\right). \tag{3.33}$$

The two boundary points $\partial A = \{(u_1, v_1, z_1), (u_2, v_2, z_2)\}$ could be written as

$$\begin{cases} u_1 = \tanh\rho_c, & u_2 = \tanh\rho_c\cos\left(\frac{2\pi x_0}{L}\right)\exp\left(i\frac{2\pi t_0}{L}\tanh\rho_c\right), \\ v_1 = 0, & v_2 = \tanh\rho_c\sin\left(\frac{2\pi x_0}{L}\right)\exp\left(i\frac{2\pi t_0}{L}\tanh\rho_c\right), \\ z_1 = \frac{1}{\cosh\rho_c}, & z_2 = \frac{1}{\cosh\rho_c}\exp\left(i\frac{2\pi t_0}{L}\tanh\rho_c\right), \end{cases} \tag{3.34}$$

and the distance of the geodesic γ_A, anchored on ∂A, could be captured by

$$\begin{aligned} L_{\gamma_A} &= R_{\rm AdS}\,\text{arcosh}\left[\frac{\cos\left(\frac{2\pi t_0}{L}\tanh\rho_c\right) + \left(\cosh^{-2}\rho_c - 1\right)\cos\left(\frac{2\pi x_0}{L}\right)}{\cosh^{-2}\rho_c}\right] \\ &= 2R_{\rm AdS}\log\left[\cosh\rho_c\sqrt{2\cos\left(\frac{2\pi t_0}{L}\right) - 2\cos\left(\frac{2\pi x_0}{L}\right)}\right] + \frac{\frac{R_{\rm AdS}}{\cosh^2\rho_c}\frac{\pi t_0}{L}\sin\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right) - \cos\left(\frac{2\pi x_0}{L}\right)} \\ &\quad + \frac{R_{\rm AdS}}{\cosh^2\rho_c}\frac{\cos\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right) - \cos\left(\frac{2\pi x_0}{L}\right)} + O\left(\frac{1}{\cosh^3\rho_c}\right) \end{aligned} \tag{3.35}$$

$$\sim 2R_{\rm AdS}\log\left[\cosh\rho_c\sqrt{2\cos\left(\frac{2\pi t_0}{L}\right) - 2\cos\left(\frac{2\pi x_0}{L}\right)}\right] + \frac{\frac{R_{\rm AdS}}{\cosh^2\rho_c}\frac{\pi t_0}{L}\sin\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right) - \cos\left(\frac{2\pi x_0}{L}\right)}. \tag{3.36}$$

Substituting equation (3.30) into equation (3.36), and identifying $\epsilon^2 = \pi c\mu/6$, one obtains

$$L_{\gamma_A} \sim 2R_{\rm AdS}\log\left[\frac{L}{2\pi\epsilon}\sqrt{2\cos\left(\frac{2\pi t_0}{L}\right) - 2\cos\left(\frac{2\pi x_0}{L}\right)}\right] + \frac{2\mu\pi^4 c_{\rm AdS}}{3L^3}\frac{R_{\rm AdS}t_0\sin\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right) - \cos\left(\frac{2\pi x_0}{L}\right)}. \tag{3.37}$$

With the RT formula, one finds the following estimation of the general entanglement entropy:

$$S_A^{(\mu)} = \frac{c_{\text{AdS}}}{3}\log\left[\frac{L}{2\pi\epsilon}\sqrt{2\cos\left(\frac{2\pi t_0}{L}\right)-2\cos\left(\frac{2\pi x_0}{L}\right)}\right]+\frac{\mu\pi^4 c_{\text{AdS}}^2}{9L^3}\frac{t_0\sin\left(\frac{2\pi t_0}{L}\right)}{\cos\left(\frac{2\pi t_0}{L}\right)-\cos\left(\frac{2\pi x_0}{L}\right)}, \tag{3.38}$$

which is a perfect match with the field theoretic result (2.16).

4 Conclusions

The $T\bar{T}$ deformation has been widely studied in recent years, due to its integrability in $2d$ CFTs and its applications in holography. In a previous work, we had amazingly found that the usual spacelike entanglement entropy alone is not sufficient to fully mark the $T\bar{T}$ deformation and that a timelike entanglement entropy must be introduced. In this paper, we proposed a general entanglement entropy that consists of both spacelike and timelike components. We demonstrated that indeed the leading $T\bar{T}$ correction to the general entanglement entropy exists in both finite size systems and finite temperature systems. We also utilized the holographic method to evaluate the general entanglement entropy. Our results show that the general entanglement entropy is necessary to fully describe the $T\bar{T}$ deformed CFT from a quantum information perspective.

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