Subregion Complementarity in AdS/CFT

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Abstract

We examine the bulk reconstruction in the AdS/CFT correspondence. We demonstrate that the subregion duality fails to hold, highlighting discrepancies between operators in causal wedge reconstruction and those in global reconstruction at the leading order in the large N limit. We argue the invalidity of the entanglement wedge reconstruction, attributing it to non-perturbative finite N effects or quantum gravity effects due to the trans-Planckian modes near the horizon. Nevertheless, we propose the subregion complementarity, illustrating that different CFT operators can describe the same bulk correlators restricted in a bulk subregion. While we expect that this complementarity is valid outside the horizon in general eternal black holes, it is inapplicable for single-sided black holes where a semi-classical description at the stretched horizon is absent.

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1   Introduction and Summary

The AdS/CFT correspondence [1] claims that a certain conformal field theory (CFT) is dual to a quantum gravity with the asymptotic AdS boundary condition. Assuming this correspondence, the CFT, called the holographic CFT, can be regarded as the definition of a quantum gravity.

The holographic CFT is typically realized by a large N gauge theory with gauge group $SU(N)$. Here, large N means $N \gg 1$, but finite, otherwise it is not a well-defined quantum field theory. On the bulk gravity side, N is related to the ratio between the Planck length and the AdS length scale, and thus N should be finite to keep both length scales finite.

The bulk semi-classical gravity description appears in the $1/N$ (asymptotic) expansion of this finite N holographic CFT around the vacuum (or another semi-classical state). In the leading order of the $1/N$ expansion around the global vacuum, we have the one-to-one correspondence between the low energy states of the holographic CFT and the free bulk states explicitly [2, 3]. In particular, for this case, the bulk local operator can be reconstructed from the CFT operator [4] based on the earlier works [5, 6].

In a quantum gravity, it is important to consider a subregion of spacetime because black hole physics is related to it. In this context, the causal wedge reconstruction [4] is anticipated to hold, where the bulk operators on the causal wedge corresponding to a CFT subregion A can be reconstructed from the CFT operators on A. It is also believed that this can be generalized to the entanglement wedge reconstruction [7] based on the agreement of the bulk and CFT relative entropy [8]. The correspondence between the reduced density matrices for bulk and CFT is also proposed [9, 10] and is called the subregion duality.

In this paper, we examine the bulk reconstruction in the AdS/CFT correspondence for the subregion. We demonstrate that the subregion duality fails to hold. Although the failure has been already claimed in [11, 12, 13, 14], we give a definite example of the violations of these in this paper. In subsection 2.2 we explicitly show discrepancies between operators in causal wedge reconstruction and those in global reconstruction even at the leading order in the large N limit. We argue the invalidity of the entanglement wedge reconstruction due to non-perturbative finite N effects or quantum gravity effects due to the trans-Planckian modes near the horizon following [13]. This is related to the fact that the generalized free theory is not a good approximation of the finite N CFT at high energy, and we have to be careful when we use the generalized free theory.

Nevertheless, we propose the subregion complementarity in section 3, illustrating that different CFT operators can describe the same bulk correlation function restricted in a bulk subregion. Indeed, the bulk local operators obtained by the AdS-Rindler HKLL bulk reconstruction [4, 15] gives the correct correlation function in the AdS-Rindler patch although they can be distinguished from the bulk local operators obtained by the global HKLL bulk reconstruction. In section 4, we argue that this complementarity is valid outside the horizon in

general eternal black holes. However, we discuss that in single-sided black holes, the complementarity cannot be applied because the semi-classical description is absent at the stretched horizon as expected from the brick-wall proposal [16, 17] and the fuzzball conjecture [18, 19]. It implies that the equivalence principle is violated near the horizon in single-sided black holes in quantum gravity.

2 Remarks on gravitational theory on subregions

2.1 Difference between bulk semi-classical gravity and finite N CFT

In this subsection, we will summarize the results obtained in the previous paper [13], emphasizing the importance of the finite N effects.

Here we first summarize the terminology of the AdS-Rindler wedge. More details of the convention of coordinates are summarized in Appendix A. Let us consider the global $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence where τ is the global time and the $\tau = \text{const.}$ surface in CFT is S^{d-1}. We can take a spherical subregion A in S^{d-1} at $\tau = 0$. The complement subregion is denoted by $\bar{A}$. We can also consider the bulk subregion M_A at bulk $\tau = 0$ slice such that M_A is enclosed by the asymptotic boundary corresponding to A and the Ryu-Takayanagi surface [20] associated with A. The bulk subregion M_A is a time slice of the AdS-Rindler patch, i.e. the intersection of $\tau = 0$ slice and the AdS-Rindler patch (see Appendix A for the details of the AdS-Rindler patch). The complement bulk subregion is $M_{\bar{A}}$. Thus, we have $M = M_A + M_{\bar{A}}$ and $S^{d-1} = A + \bar{A}$, where M is the bulk $\tau = 0$ slice in the global AdS_{d+1}, and S^{d-1} is the $\tau = 0$ slice of the cylinder $\mathbf{R} \times S^{d-1}$ on which the CFT_d lives. We will denote the (union of future and past) domain of dependence, which is sometimes called the causal diamond, of A as $D(A)$ and the one of M_A as $D(M_A)$ which is the region that the AdS-Rindler patch covers, and we call $D(A)$ and $D(M_A)$ the Rindler wedge and the AdS-Rindler wedge respectively. In this case, the entanglement wedge is the same as the causal wedge, and we do not distinguish them.

In the global $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence, $1/N$ expansion in the large N limit of the holographic CFT corresponds to the perturbative expansion around the global (empty) AdS in the low energy. Our standard notion of bulk spacetime is only valid in this semi-classical approximation, although the true quantum gravity is defined by the finite N CFT. Note that, in this paper, words like the gravity theory or the bulk theory mean this semi-classical theory.

Since we have decomposed the bulk and CFT $\tau = 0$ slices as $M = M_A + M_{\bar{A}}$ and $S^{d-1} = A + \bar{A}$, we can consider theories on subregions. The question is whether the semi-classical gravitational theory for the AdS-Rindler observer is dual to the CFT defined on the Rindler wedge.

Bulk AdS-Rindler At the leading order of the semi-classical expansion, the bulk theory is a free theory on empty AdS_{d+1} spacetime, and the quantization of the free theory can be done. In the AdS-Rindler wedge, we can take the AdS-Rindler time, and can take the creation/annihilation operators associated with this choice of time. As for the usual discussions on the Unruh effects, we can argue that the combined system of the free quantum theories on the AdS-Rindler patches $D(M_A)$ and $D(M_{\bar{A}})$ is complete and there is the Bogoliubov transformation between the creation/annihilation operators of this system and those for the global time on the whole spacetime.[1] In particular, the bulk local operators $\phi^R(X)$ expressed by the AdS-Rindler modes in $D(M_A)$ is identical to the ones $\phi(X)$ at the same points expressed by the global modes. Here we put the superscript R to the operators as $\phi^R(X)$ in order to distinguish the operators act on the bulk Hilbert space obtained by the AdS-Rindler quantization from that by the global time quantization although they are the same in the AdS-Rindler wedge in the (UV complete) bulk (free) theory.

CFT Rindler In the CFT picture, it is known [21] that the causal diamonds $D(A)$ of the spherical subregion A in the whole spacetime $\mathbf{R}_\tau \times S^{d-1}$ is conformal to $\mathbf{R}_{t_R} \times \mathbf{H}^{d-1}$ where t_R is the Rindler time (the explicit map is given by (A.8) in Appendix A). Thus, the CFT on the Rindler wedge $D(A)$ is just the CFT on $\mathbf{R} \times \mathbf{H}^{d-1}$. In particular, for $d = 2$ it is the CFT on the Minkowski spacetime $\mathbf{R}^{1,1}$.

A paradox in AdS/CFT The above bulk and CFT descriptions can be considered independently and both are correct. In AdS/CFT, the bulk picture is equivalent to the low energy (and large N) approximation of CFT. The operators in these two descriptions are related, by the BDHM relation [5], which claims that the bulk field $\phi(X)$ becomes the CFT primary field $\mathcal{O}^{\mathrm{CFT}}(x)$ by bringing it to the boundary with an appropriate scaling factor. We assume the BDHM relation for the global AdS/CFT correspondence. We now consider the bulk free field $\phi^R(X)$ on the bulk subregion M_A. We will bring it to the boundary and denote it by $\mathcal{O}(x)$ with an appropriate factor. Note that this bulk field $\phi^R(X)$ is equivalent to the bulk field $\phi(X)$ defined in the global AdS as stated above. Thus, according to the BDHM relation, $\mathcal{O}(x)$ should become the CFT primary field $\mathcal{O}^{\mathrm{CFT}}(x)$ on A (up to a scaling factor). In particular for $d = 2$, the CFT on A is that on Minkowski spacetime. However, the $\mathcal{O}(x)$ cannot be $\mathcal{O}^{\mathrm{CFT}}(x)$ because $\mathcal{O}(x)$ contains the creation operators associated with modes $e^{-i\omega t + ipx}$ with $\omega^2 - p^2 < 0$ and such tachyonic operators[2] cannot exist. Note that the energy of the tachyonic creation operators can be negative by the Lorentz transformation. This seems to be a paradox [13].

The resolution Here, it is important to note that the bulk free theory, which is the generalized free theory in the CFT picture, is only the low energy and large N approximation of the

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[1]The vacuum for the Rindler time is different from the global vacuum.

[2]In the bulk theory in the AdS-Rindler wedge, these operators are not tachyonic.

(finite N) CFT. The bulk free theory on the AdS-Rindler patch $D(M_A)$ is a good description only for the low-energy sectors of the free theory on the global patch. However, the low-energy sector for the AdS-Rindler modes is not the same as that for the global modes [13]. The free theory on the AdS-Rindler patch is invalid as an approximation of the finite N CFT, which is regarded as the quantum gravity, even though the bulk free theory, which is not dual to CFT, on the AdS-Rindler patch itself is consistent if we assume that it is the UV complete theory. Thus, the paradox is resolved.

A general lesson here from the above discussion is that the bulk semi-classical gravitational theory as the low-energy effective theory of quantum gravity (= the finite N CFT) is observer-dependent. Specifically, its validity depends on the chosen patch and its associated time foliation. When we consider a causal diamond of a spacetime subregion, like the AdS-Rindler wedge $D(M_A)$, we encounter "horizons" corresponding to the boundary of the causal diamond. A semi-classical gravitational theory formulated within these causal diamonds, accompanied by a time foliation that causes an innumerable number of time slices to converge near the horizon (akin to the Rindler time), becomes unreliable near the horizon as the approximation of the finite N CFT. This unreliability is attributed to the UV cut-off, typically associated with the Planck mass, of the bulk effective theory. We stress that the breakdown of the effective theory can be seen by examining the finite N effects. This is because the $1/N$ expansion (i.e. the semi-classical expansion) is based on the leading-order spectrum which is obtained by taking $N = \infty$. In this sense, this is the non-perturbative quantum gravity effect. The true theory is quantum gravity defined by the finite N CFT, and the semi-classical gravity obtained by the perturbative $1/N$ expansion around $N = \infty$ is just an effective description. For large but finite N, we need non-perturbative effects which especially cannot be negligible near the "horizon" for an observer in the subregion.

We call the semi-classical theory for the observer in a subregion (i.e. theory with the patch and time-foliation inside a causal diamond) the bulk theory on the subregion. The statement that the bulk theory on the subregion is inapplicable might be surprising. This is because the semi-classical expansion of the bulk gravity theory is anticipated to remain valid even within spacetime subregions, such as the Rindler region or the more general causal diamonds. Such an approach has been employed extensively in numerous studies. For instance, the AdS-Rindler wedge reconstruction—a specific case of the entanglement wedge construction—relies on this very assumption. The subregion duality and other topics that use the entanglement wedge reconstruction are rooted in this potentially flawed presumption.[3] In the following subsection, we will demonstrate how the subregion duality and the entanglement wedge reconstruction do not work. We here remark that the subregion duality may hold between the bulk free theory and the generalized free theory (or extension of them including $1/N$ perturbative corrections). Nevertheless, the generalized free theory is not a good approximation of a finite N CFT at

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[3]If we proceed with the wrong assumption, we will face pathological things. Historically, such pathological things have been pretended not to exist by assuming that the holographic error correction code proposal is correct [22]. This proposal was made just to avoid the "pathology" and is unnecessary.

high energy and we must take account of the non-perturbative corrections.

On the other hand, the invalidity of the semi-classical description near the horizon (or at the stretched horizon) is known as the trans-Planckian problem [16]. This is due to the use of the coordinates which are singular at the horizon. It may imply that UV physics is important near the horizon and cannot be neglected as suggested by the brick wall model [16] or the fuzzball conjecture [18, 19]. Here, we note that in order to discuss the bulk reconstruction or the entanglement entropy for a subregion, the full (entanglement) spectrum of the subregion is needed. It means that we need the whole subregion including the vicinity of the horizon.[4]

2.2 Failure of the subregion duality and entanglement wedge reconstruction

In this subsection, we explain in what sense the subregion duality and the causal (and also entanglement) wedge reconstruction do not work.

The subregion duality is the claim that for any states ρ, σ (around the global vacuum), we have

$$\rho_A = \sigma_A \iff \rho_{M_A} = \sigma_{M_A}, \tag{2.1}$$

for a subregion A for the CFT and the bulk subregion M_A, which is a time-slice of the entanglement wedge of A, for the bulk theory where $\rho_A = \mathrm{tr}_{\bar A}(\rho)$ and $\rho_{M_A} = \mathrm{tr}_{M_{\bar A}}(\rho)$. Note that the low-energy Hilbert space $\mathcal{H}$ is the same in CFT and the bulk theory, and ρ, σ are the density matrices in this space. $\mathcal{H}$ is formally decomposed into $\mathcal{H}_A \otimes \mathcal{H}_{\bar A}$ in the CFT side, but this decomposition is in principle not directly related to the decomposition in the bulk side $\mathcal{H}_{M_A} \otimes \mathcal{H}_{M_{\bar A}}$.

The claim (2.1) is equivalent to that the relative entropies are the same:

$$S(\rho_A | \sigma_A) = S(\rho_{M_A} | \sigma_{M_A}), \tag{2.2}$$

which was shown in [8] by using the semi-classical bulk computations [23]. The entanglement wedge reconstruction straightforwardly follows from the subregion duality (2.1) [7]. It claims that for any (low-energy) bulk operator ϕ_{M_A} supported in M_A there exists the CFT operator $\mathcal{O}_A$ supported in A such that

$$\phi_{M_A} |\psi\rangle = \mathcal{O}_A |\psi\rangle, \tag{2.3}$$

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[4]If the smeared bulk local operator in the AdS-Rindler patch is not close to the horizon, it contains almost the low energy modes only. Here, the smearing is defined by essentially the Gaussian in the AdS-Rindler coordinate (See Appendix C).

in the low energy approximation,[5] where $|\psi\rangle$ are any states in the low energy subspace around the global vacuum $|0\rangle$. Via the global bulk reconstruction [4], ϕ_{M_A} is a CFT operator supported on the entire space S^{d-1}. Thus, (2.3) claims that there exists a CFT operator $\mathcal{O}_A$ supported on subregion A for a CFT operator $\mathcal{O}$ supported on the whole space, such that $\mathcal{O}_A$ acts on $|\psi\rangle$ as $\mathcal{O}$ does. Here, it is important that the reconstructed operator $\mathcal{O}_A$ does not depend on $|\psi\rangle$, like the HKLL reconstruction. This also implies that if $\mathcal{O}_A^1, \mathcal{O}_A^2$ are CFT operators corresponding to bulk low-energy operators $\phi_{M_A}^1, \phi_{M_A}^2$, the CFT operator corresponding to the product operator $\phi_{M_A}^1 \phi_{M_A}^2$ must be $\mathcal{O}_A^1 \mathcal{O}_A^2$ because $|\psi'\rangle \equiv \phi_{M_A}^2 |\psi\rangle = \mathcal{O}_A^2 |\psi\rangle$ is also in the low-energy subspace. Thus, this is the operator reconstruction (in the subspace) and not the state reconstruction. Indeed, via the Reeh–Schlieder theorem, we can reconstruct arbitrary states from operators in any subregion not limited to the entanglement wedge by acting them on the vacuum, and the notion of the entanglement wedge reconstruction becomes non-sense.

It is important to note that the "derivation" of the subregion duality and entanglement wedge reconstruction is based on the semi-classical expansion. The leading part of the expansion is described by the generalized free theory. As we stated above, the generalized free theory is different from the finite N CFT at high energy. Entanglement entropy is a UV-dependent quantity, and thus there is no reason that the entropy in the generalized free theory is the same as that in the finite N CFT, even if we compare the universal terms of the entropy. For instance in [23], the bulk entanglement entropy is computed based on the semi-classical expansion where the 1-loop computations are based on the generalized free approximation. The bulk modular Hamiltonian (for bulk free fields) is the modular Hamiltonian for the generalized free on the boundary theory which is not that for the finite N CFT. Indeed, the bulk modular Hamiltonian (which is the AdS-Rindler Hamiltonian for the AdS-Rindler case) contains arbitrary high energy modes for the global Hamiltonian [13] and we cannot trust it because it is beyond the low-energy approximation. To compute the bulk entanglement entropy, we need information at the short distance where we expect the non-perturbative quantum gravity effects to be non-negligible. As stressed above, the bulk theory on a subregion cannot be regarded as the low energy approximation of the CFT for the region near the horizon (entangling surface caused by taking the subregion), while the vicinity of the entangling surface dominantly contributes to the entanglement entropy. Thus, the discussion based on the bulk entanglement (i.e. entanglement for generalized free theory) is not reliable for the approximation of the finite N CFT.

When we treat the semi-classical bulk theory as a UV complete theory, the subregion duality and entanglement wedge reconstruction are applicable (see e.g. [24, 25]), that is, they hold, at least for the smeared low-energy operators not close to the horizon of the causal diamond, in the generalized free theory, not the full CFT, although high-energy modes appear near the horizon (see Appendix C). On the other hand, when we view the bulk theory as the low-energy approximation of the CFT, we encounter issues with both conformal symmetry

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[5]More precisely, the equality in (2.2) and (2.3) should be understood as the approximate equality $\simeq$ in the low energy and large N approximation. We will use $=$ instead of $\simeq$ just for simplicity.

and unitarity. Such contradictions were already found as the radial locality paradox in [22], and it was assumed to be resolved by the existence of the quantum error correction code-like structure. However, we assert that such a structure is non-existent because all paradoxes come from the inappropriate use of the generalized free description.

Below, we will explicitly demonstrate the invalidity of the subregion duality and the entanglement wedge reconstruction, by showing a case that both sides of eq. (2.3) are different even at the leading order of the large N limit. We here take a spherical subregion A for the CFT defined on the sphere as in the previous subsection. Then, the entanglement wedge (causal wedge) in the bulk is the AdS-Rindler patch in the global AdS, i.e., the associated bulk subregion is M_A defined in the previous subsection. We now consider smeared bulk local operators so that our discussion is closed in the low-energy sector as

$$\tilde{\phi}_{M_A} = \int dX' K(X')\phi(X'), \tag{2.4}$$

where $K(X')$ is a smearing function (or distribution) supported only on bulk subregion M_A. Do not confuse this smearing with smearing used in the HKLL reconstruction. Here the smearing function K in (2.4) is introduced just to consider a low-energy operator and is arbitrary if it is a smooth function and its support is large enough such that it contains mostly low energy modes. Via the global HKLL reconstruction [4], $\phi(X')$ is a CFT operator supported on the entire region S^{d-1}, and thus $\tilde{\phi}_{M_A}$ is a smooth function such that it contains mostly low energy modes. We consider the following state given by the unitary operation:

$$|K\rangle = e^{i\epsilon\tilde{\phi}_{M_A}} |0\rangle, \tag{2.5}$$

where the smearing function K was chosen to be real and ϵ is a real small parameter and thus $e^{i\epsilon\tilde{\phi}_{M_A}}$ is unitary. Let ρ, σ be the density matrix for $|K\rangle$ and $|0\rangle$ as $\rho = |K\rangle\langle K|, \sigma = |0\rangle\langle 0|$. Note that we cannot distinguish $|K\rangle$ with the global vacuum in $M_{\bar{A}}$, i.e. $\rho_{M_{\bar{A}}} = \sigma_{M_{\bar{A}}}$ where $\rho_{M_{\bar{A}}} \equiv \mathrm{tr}_{M_A} \rho, \sigma_{M_{\bar{A}}} \equiv \mathrm{tr}_{M_A} \sigma$ because $e^{i\epsilon\tilde{\phi}_{M_A}}$ is a unitary operator acting only on $\mathcal{H}_{M_A}$. Let us take the CFT subregion A bigger than the half of the whole space S^{d-1} and consider the smearing function spherically symmetric around the center of AdS space and supported on a small region such that the smearing function K is supported on M_A although it is much larger than the cut-off length (see Figure 1).[6] Then, it should have a spherically symmetric energy density in the CFT picture. Using the global HKLL bulk reconstruction, this state is given by

$$|K\rangle = e^{i\epsilon \int d\tau \tilde{K}(\tau) \int d\Omega\, \mathcal{O}(\tau,\Omega)} |0\rangle, \tag{2.6}$$

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[6]If we want to take A as the half space of the entire sphere, we can move the bulk local operator slightly toward M_A by the conformal transformation such that the bulk operator ϕ_{M_A} is supported on M_A. It should have a non-zero energy density for any point, in the CFT picture, by the continuity. The discussions below can be valid for this modification.

where $\tilde{K}(\tau)$ is a function of τ depending on K, but Ω-integration is spherically symmetric. Then, we find $\rho_{\bar{A}} \neq \sigma_{\bar{A}}$ because the unitary operator acts on the entire region including $\bar{A}$ for $|K\rangle$ in (2.6). Thus, the subregion duality (2.1) is violated where $\sigma = |0\rangle\langle 0|$. (The roles of A and $\bar{A}$ are reversed.)
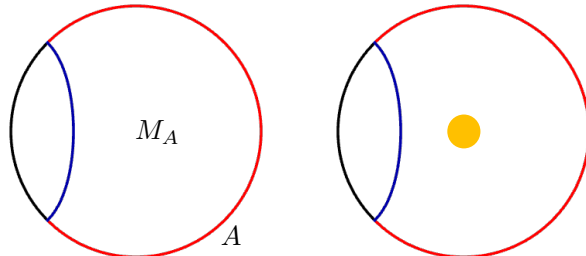

Figure 1: AdS-Rinlder subregion. In the left figure, M_A is the bulk subregion enclosed by the boundary subregion A (red curve) and the Ryu-Takayanagi surface (blue curve). The smearing function K has a support (orange region in the right figure) which is spherically symmetric and included in M_A.

The entanglement wedge reconstruction (2.3), which is now the causal wedge reconstruction, requires that there exists the state

$$|K^R\rangle = e^{i\epsilon \tilde{\phi}^R_{M_A}} |0\rangle , \tag{2.7}$$

such that $\tilde{\phi}^R_{M_A}$ is constructed from the CFT operators only supported on A such that $|K\rangle = |K^R\rangle$.[7] Let us confirm that there exist no such operators $\tilde{\phi}^R_{M_A}$ by comparing the energy density $T_{00}(\tau = 0, x)$ at x for the two states $|K\rangle$ and $|K^R\rangle$ in the CFT picture. Indeed, generically,

$$\langle K| T_{00}(\tau = 0, x) |K\rangle \neq \langle K^R| T_{00}(\tau = 0, x) |K^R\rangle , \tag{2.8}$$

for the boundary point x space-like separated from all points of the causal diamonds of A, for example, x in $\bar{A}$ (at $\tau = 0$). Because of the spherical symmetry of $|K\rangle$, it is clear that

$$\langle K| T_{00}(\tau = 0, x) |K\rangle = \frac{1}{\mathrm{Vol}(S^d)} \langle K| H |K\rangle = \frac{1}{\mathrm{Vol}(S^d)}(\epsilon^2 \langle 0| \tilde{\phi}_{M_A} H \tilde{\phi}_{M_A} |0\rangle + \mathcal{O}(\epsilon^3)), \tag{2.9}$$

where H is the Hamiltonian, is non-zero and $\mathcal{O}(\epsilon^2)$. Here, we subtracted the Casimir energy

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[7]Here, we note again that the bulk operator $\tilde{\phi}$, not just the state $\tilde{\phi}^n |0\rangle$, should be reconstructed. Indeed, the global and AdS-Rindler HKLL bulk reconstructions give such an operator and we will see later that the reconstructed CFT operators give correct bulk correlation functions even though the subregion duality and the entanglement wedge reconstruction are not valid. If we just want to reconstruct the states, the CFT operators on A, for the Rindler wedge case, can give any CFT (and bulk) state and the notion of entanglement wedge reconstruction is useless. This is because using the mirror operator for the thermofield double state (see Appendix B), we can obtain $\mathcal{O}_{\bar{A}} |0\rangle$, where $\mathcal{O}_{\bar{A}}$ is any CFT operator supported on $\bar{A}$ by $\mathcal{O}_A |0\rangle$, where $\mathcal{O}_A$ is a CFT operator supported on A. More generally, via the Reeh–Schlieder theorem, we can reconstruct arbitrary states from any subregion not limited to the entanglement wedge.

from the definition of T_{00} and the Hamiltonian such that $\langle 0|\, T_{00}(\tau = 0, x)\, |0\rangle = 0$. On the other hand,

$$\langle K^R|\, T_{00}(\tau = 0, x)\, |K^R\rangle$$
$$= \langle 0|\, (T_{00}(\tau = 0, x) + [-i\epsilon\tilde{\phi}^R_{M_A}, T_{00}(\tau = 0, x)] + \cdots)\, |0\rangle = 0, \qquad (2.10)$$

exactly by the causality of CFT.[8] Thus, $|K\rangle$ and $|K^R\rangle$ cannot be the same states and thus the entanglement wedge reconstruction (2.3) does not hold. Note that we have used only the three-point functions (single T and two $\mathcal{O}$) for $\mathcal{O}(\epsilon^2)$. Therefore, (2.3) is invalid even at the leading order in the large N expansion.

We can repeat this discussion for $|K\rangle = \sum_{n=0}^{n_0} \frac{1}{n!}(i\epsilon\tilde{\phi}_{M_A})^n\, |0\rangle$ and $|K^R\rangle = \sum_{n=0}^{n_0} \frac{1}{n!}(i\epsilon\tilde{\phi}^R_{M_A})^n\, |0\rangle$, which are the polynomials of the bulk local fields. Then, the energy density for $|K\rangle$ is uniform and approximately $\langle 0|\, (\epsilon\tilde{\phi}_{M_A})T_{00}(\epsilon\tilde{\phi}_{M_A})\, |0\rangle$ which is $\mathcal{O}(\epsilon^2)$. On the other hand, by the causality of CFT we can show $\langle K^R|\, T_{00}(\tau = 0, x)\, |K^R\rangle = \langle 0|\, (T_{00}(\tau = 0, x) + [-i\epsilon\tilde{\phi}^R_{M_A}, T_{00}(\tau = 0, x)] + \cdots)\, |0\rangle = \mathcal{O}(\epsilon^{n_0+1})$, thus $\langle K|\, T_{00}(\tau = 0, x)\, |K\rangle \neq \langle K^R|\, T_{00}(\tau = 0, x)\, |K^R\rangle$.

Contradiction with the quantum corrected Ryu-Takayanagi formula We can also argue that the entanglement wedge reconstruction contradicts the FLM proposal [23] that the CFT entanglement entropy for the subregion A agrees with the area term (Ryu-Takayanagi formula) including the back reaction and the bulk entanglement entropy for the $\mathcal{O}(N^0)$ order in the $1/N$ expansion. If we consider an excited state $|\Psi\rangle$ where the expectation value of the CFT stress-energy tensor $\langle\Psi|\, T_{\mu\nu}\, |\Psi\rangle = \mathcal{O}(N^0)$ and compare the difference of the entropy between $|\Psi\rangle$ and the vacuum state $|0\rangle$, the FLM proposal means that

$$\delta S_A^{\mathrm{CFT}} = \frac{\delta A_\Sigma}{4G_N} + \delta S_{M_A}^{\mathrm{bulk}} \qquad (2.11)$$

holds for the leading order of the $1/N$ expansion, where $\delta S_A^{\mathrm{CFT}}$ is the difference between the entanglement entropy for $|\Psi\rangle$ and $|0\rangle$ for subregion A in CFT, $\delta S_{M_A}^{\mathrm{bulk}}$ is that for bulk subregion M_A, and δA_Σ is the change of the area of the Ryu-Takayanagi surface Σ for the subregion A due to the back reaction by the excitation. Note that the entanglement entropy itself is UV divergent but the differences between the excited states and the vacuum are UV finite.

As an excited state $|\Psi\rangle$, let us take the bulk coherent state $|\Psi\rangle = e^B\, |0\rangle$ for the free bulk scalar field corresponding to an arbitrary classical configuration ϕ_{cl} (see Appendix D for details). The operator e^B is unitary and also factorized in the bulk as $e^B = e^{B_{M_A}} e^{B_{M_{\bar{A}}}}$ where B_{M_A} and $B_{M_{\bar{A}}}$ are supported only on the bulk subregion M_A and $M_{\bar{A}}$ respectively. Then, the bulk entanglement entropy for $|\Psi\rangle$ is the same as that for the vacuum and $\delta S_{M_A}^{\mathrm{bulk}} = 0$. The area term δA_Σ is computed by solving the Einstein equation including the contribution of

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[8]Note that without changing the discussion here, we can smear $T_{00}(\tau = 0, x)$ appropriately in order to consider the low energy modes only. Thus, even in the low energy approximation, (3.11) does not hold.

the classical bulk stress tensor $T_{\mu\nu}^{\text{bulk}}[\phi_{cl}]$ corresponding to the classical configuration ϕ_{cl} which is computed as $T_{MN}^{\text{bulk}}[\phi_{cl}] = \langle\Psi| T_{MN}^{\text{bulk}} |\Psi\rangle - \langle 0| T_{MN}^{\text{bulk}} |0\rangle$. Since the classical configuration is arbitrary, we expect that there are cases where the back-reaction deforms the area of the Ryu-Takayanagi surface $\delta A_\Sigma \neq 0$. Note that the correction δA_Σ is $\mathcal{O}(G_N)$ and thus $\delta A_\Sigma/(4 G_N)$ is $\mathcal{O}(G_N^0)$ or $\mathcal{O}(N^0)$. Thus, the right-hand side of (2.11) can be non-vanishing values at the leading order $\mathcal{O}(N^0)$.

On the other hand, if the entanglement wedge construction holds, the bulk operators B_{M_A} and $B_{M_{\bar{A}}}$ can be represented as CFT operators supported on A and $\bar{A}$ respectively. Thus, the operators $e^{B_{M_A}}$ and $e^{B_{M_A}}$ are the unitary operators on A and $\bar{A}$. It leads to $\delta S_A^{\text{CFT}} = 0$ that contradicts (2.11). Therefore, the entanglement wedge is incompatible with the FLM proposal. As we stressed in the previous subsection, we may encounter many contradictions when we consider subregions if we trust the $1/N$ expansion (or the generalized free theory with the higher-order $1/N$ perturbative corrections).

A simple bulk reconstruction picture   As we will see below, we can identify which part of the bulk local operator supported on M_A cannot be reconstructed from the CFT operators supported on A as done in [11, 12, 14] using the results in [3] [26]. In these papers, the bulk wave packet state was constructed from the (smeared) bulk local operators. Because it always reaches the asymptotic boundary by the backward time-evolution and the BDHM relation in the global patch, it is represented by the time-evolution of the states given by acting smeared CFT primary operator with almost fixed energy and momentum on the vacuum at a point on the boundary, say x. Although the wave packet is localized on a (light-like) curve in the bulk, the time evolution of the state in the CFT picture is similar to the two light-like particles (in the two-dimensional CFT) and the VEV of the energy-momentum tensor is localized around the two points on a time slice. Thus, if $x \in D(A)$, this state is supported in A because the two points stay in $D(A)$. However, if $x \notin D(A)$ the state is not supported in A, and the bulk wave packet localized on the geodesics (that we called the horizon-to-horizon geodesics in [13]) leaving on the past horizon and passing through the future horizon in the AdS-Rindler patch $D(M_A)$. For the states corresponding to the wave packets localized on the horizon-to-horizon geodesics, (2.3) does not hold.[9] Note that the dominant modes for the wave-packet localized on the horizon-to-horizon modes correspond to the "tachyonic" modes for the CFT on the Rindler wedge $D(A)$ which should be absent in the finite N CFT [13].

Let us consider why the global and AdS-Rindler HKLL bulk reconstructions are different. If we regard the bulk gravity theory as the UV complete theory,

$$\langle 0| \phi(X)\phi(X') |0\rangle = \text{tr}_{M_A}(\rho_{M_A}\phi^R(X)\phi^R(X')), \tag{2.12}$$

where $\phi^R(X)$ is constructed from the mode expansion in the right AdS-Ridnler wedge, will

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[9]More precisely, if we consider (2.7) where $\tilde{\phi}^R$ is the CFT operator for the bulk wave packet, the violation is clear because the VEV of the energy-momentum tensor at the point outside $D(A)$ should vanish.

hold as we can see from the usual discussion on the Unruh effect in Minkowski space. If X, X' are not close to the AdS-Rindler horizon, (2.12) approximately holds for the gravity dual of the finite N holographic CFT as explained in the next section. However, it does not hold if X, X' are close to the AdS-Rindler horizon (see Appendix C). The HKLL bulk reconstructions use the equations of motion of the bulk theory in order to relate the bulk local field and CFT operators. However, the equations of motion in the bulk low energy description are not justified near the horizon although we need them to relate the global and AdS-Rindler modes. It is manifest if we consider the horizon-to-horizon bulk wave packets because the semi-classical description for the AdS-Rindler observer is not valid at the stretched horizon. Thus, we cannot trust the relation between the global and AdS-Rindler modes when the bulk free description is not UV complete.

3   AdS/CFT for subregion

In the AdS/CFT correspondence, it is expected that if we take (at least for a simple) subregion A of CFT, the gravitational theory on a subregion M_A, whose boundary consists of A and the Ryu-Takayanagi surface of A, in the bulk is dual to the CFT on A. However, we have seen that the subregion duality and the causal (and also entanglement) wedge reconstruction are not valid. Thus the above expectation seems to be invalid. Nevertheless, the AdS/CFT correspondence for the subregion may be correct in the following sense at least for the AdS-Rindler case: Any (global) vacuum correlation functions of the low-energy operators supported only in M_A can be reproduced from the CFT correlation functions on A. We will show it in two different ways: using the bulk picture and AdS/CFT correspondence for the Poincare patch.

Let us consider the vacuum state $\sigma = |0\rangle \langle 0|$ in the CFT on S^{d-1} with the global Hamiltonian. For the Rindler subregion A, the reduced density matrix is $\sigma_A = \text{tr}_{\bar{A}} \sigma$. In the bulk picture, the vacuum state $\sigma = |0\rangle \langle 0|$ is as the global AdS vacuum $|0\rangle \langle 0|$. We can consider the corresponding bulk subregion M_A and define the reduced density matrix on M_A as $\sigma_{M_A} = \text{tr}_{\bar{M}_A} \sigma$. (This partial trace is not well-defined because the bulk spacetime description is only an approximation, and then σ_{M_A} makes sense only for the low energy approximation.)

The above statement of the AdS/CFT correspondence for subregion is that for any (smeared) bulk local operator $\phi(X)$ on the subregion ($X \in D(M_A)$) there exists the CFT operator $\phi^R(X, \mathcal{O}^{\text{CFT}})$ supported only in CFT subregion A such that we have

$$\langle 0| \phi(X_1) \ldots \phi(X_n) |0\rangle = \text{tr}_A(\sigma_A \phi^R(X_1, \mathcal{O}^{\text{CFT}}) \ldots \phi^R(X_n, \mathcal{O}^{\text{CFT}})) \qquad (3.1)$$

where all points X_i are in $D(M_A)$ except for the stretched horizon region (neighborhood of the boundary of $D(M_A)$). Note that $\phi^R(X, \mathcal{O}^{\text{CFT}})$ is different from the global HKLL reconstruction of the bulk operator $\phi(X)$ that we denote by $\phi^G(X, \mathcal{O}^{\text{CFT}})$ whose support is the entire space

S^{d-1}. As we stated in the previous subsection, the subregion duality does not hold and thus

$$\phi^G(X, \mathcal{O}^{\text{CFT}}) |0\rangle \neq \phi^R(X, \mathcal{O}^{\text{CFT}}) |0\rangle \tag{3.2}$$

although the correlation functions are the same as

$$\begin{aligned}
\langle 0| \phi^G(X_1, \mathcal{O}^{\text{CFT}}) \ldots \phi^G(X_n, \mathcal{O}^{\text{CFT}}) |0\rangle &= \langle 0| \phi(X_1) \ldots \phi(X_n) |0\rangle \\
&= \langle 0| \phi^R(X_1, \mathcal{O}^{\text{CFT}}) \ldots \phi^R(X_n, \mathcal{O}^{\text{CFT}}) |0\rangle
\end{aligned} \tag{3.3}$$

for arbitrary $X_i \in D(M_A)$ except for the stretched horizon region. Both states in (3.2) are different in the entire space as can be confirmed by looking at the energy distribution as done in the previous section. However, the two CFT operators $\phi^G(X_1, \mathcal{O}^{\text{CFT}})$ and $\phi^R(X_1, \mathcal{O}^{\text{CFT}})$ describe the same physics in the bulk subregion M_A. This sounds similar to the black hole complementarity [27] in the sense that there are different descriptions for the same physics in the subregion (the outside of the "horizon"). We call the concept the subregion complementarity.

3.1 Using bulk picture

First, we will consider the free limit of the bulk theory as the UV complete theory although only at the low energy the semi-classical theory is a good approximation of the finite N holographic CFT. Then, as explained above, the bulk local operator[10] $\phi^R(X)$ in the AdS-Rindler quantization in the AdS-Rindler subregion $D(M_A)$ is identical to the bulk local operator $\phi(X)$ in the global-time quantization at the same point in the subregion in the entire spacetime. This implies that the correlation functions are the same, i.e.

$$\langle 0| \phi(X)\phi(X') |0\rangle = \text{tr}_{M_A}(\sigma_{M_A} \phi^R(X)\phi^R(X')), \tag{3.4}$$

where $X, X' \in D(M_A)$. Then, if we define $\mathcal{O}(x)$ as the boundary limit of $\phi(X)(= \phi^R(X))$ with an appropriate scaling factor, $\mathcal{O}(x)$ gives the correct two-point function of the corresponding scalar of CFT. Here, $\phi^R(X)$ is written by the AdS-Rindler creation/annihilation modes and $\mathcal{O}(x)$ can be written by them also. $\phi(X)$ is written by the global AdS creation/annihilation modes and $\mathcal{O}(x)$ can be written by them also. Using the global AdS modes, we can rewrite $\phi(X)$ as a functional of $\mathcal{O}(x)$, which is given, for example, by the global HKLL bulk reconstruction formula, and we will denote it as $\phi^G(X, \mathcal{O})$. On the other hand, using the AdS-Rindler modes, we can rewrite $\phi^R(X)$ as a functional of $\mathcal{O}(x)$, which is given, for example, by the AdS-Rindler HKLL bulk reconstruction formula, and we will denote it as $\phi^R(X, \mathcal{O})$. Note that $\phi^R(X, \mathcal{O})$ is linear in $\mathcal{O}(x)$ and the support is in A. Thus, we have

$$\text{tr}_{M_A}(\sigma_{M_A} \phi^R(X)\phi^R(X')) = \langle 0| \phi^R(X, \mathcal{O})\phi^R(X', \mathcal{O}) |0\rangle = \text{tr}_A(\sigma_A \phi^R(X, \mathcal{O})\phi^R(X', \mathcal{O})). \tag{3.5}$$

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[10]More precisely we here consider the smeared operators so that they contain only low-energy modes, and the smearing functions are supported only in M_A.

Because the vacuum two-point function is determined only by the conformal dimension, we should have

$$\text{tr}_A(\sigma_A \phi^R(X, \mathcal{O})\phi^R(X', \mathcal{O})) = \text{tr}_A(\sigma_A \phi^R(X, \mathcal{O}^{\text{CFT}})\phi^R(X', \mathcal{O}^{\text{CFT}})) \quad (3.6)$$

where $\mathcal{O}^{\text{CFT}}(x)$ is the primary scalar operator of the finite N CFT.[11] Thus, for the two-point function at $X, X' \in D(M_A)$, we have

$$\langle 0 | \phi(X)\phi(X') | 0 \rangle = \text{tr}_A(\sigma_A \phi^R(X, \mathcal{O}^{\text{CFT}})\phi^R(X', \mathcal{O}^{\text{CFT}})). \quad (3.7)$$

This also means that the bulk correlation functions can be reproduced both from the global and the Rindler reconstructions:

$$\langle 0 | \phi(X)\phi(X') | 0 \rangle = \langle 0 | \phi^G(X, \mathcal{O}^{\text{CFT}})\phi^G(X', \mathcal{O}^{\text{CFT}}) | 0 \rangle = \langle 0 | \phi^R(X, \mathcal{O}^{\text{CFT}})\phi^R(X', \mathcal{O}^{\text{CFT}}) | 0 \rangle . \quad (3.8)$$

Multi-point correlation functions can also be reconstructed in the large N and low-energy limit because at the leading order the bulk theory is free and the large N factorization holds in the CFT side. Therefore, in the sense that we can reproduce the bulk correlation functions on the global vacuum, the AdS-Rindler HKLL bulk reconstruction formula is correct and the AdS/CFT correspondence for the subregion holds.

It is important to note that, even though (3.7) is correct, ϕ^G and ϕ^R are different CFT operators as

$$\phi^G(X, \mathcal{O}^{\text{CFT}}) | 0 \rangle \neq \phi^R(X, \mathcal{O}^{\text{CFT}}) | 0 \rangle \quad (3.9)$$

even in the low energy approximation as we remarked above. More precisely, for the smeared bulk local operators defined as

$$\tilde{\phi}^G(\mathcal{O}^{\text{CFT}}) = \int dX' K(X')\phi^G(X', \mathcal{O}^{\text{CFT}}), \quad \tilde{\phi}^R(\mathcal{O}^{\text{CFT}}) = \int dX' K(X')\phi^R(X', \mathcal{O}^{\text{CFT}}), \quad (3.10)$$

where $K(X')$ is a smearing function (or distribution) supported on M_A,

$$\tilde{\phi}^G(\mathcal{O}^{\text{CFT}}) | 0 \rangle = \tilde{\phi}^R(\mathcal{O}^{\text{CFT}}) | 0 \rangle , \quad (3.11)$$

does not hold as we saw in the previous section. We remark that the width of the smearing function $K(X')$ must be much greater than the cut-off length (i.e., the Planck length) and thus the center of $K(X')$ is further away from the AdS-Rindler horizon than the Planck length,

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[11]The AdS-Rindler modes contain "tachyonic" creation/annihilation modes. For $\phi^R(X, \mathcal{O}^{\text{CFT}})$, these correspond to the Fourier transformation of $\mathcal{O}^{\text{CFT}}(x)$ with the tachyonic energy momenta in the Rindler coordinates on A. Such Fourier modes $\mathcal{O}^{\text{CFT}}(x)$ vanish around the Rindler vacuum, but do not vanish around the finite temperature state.

and (3.8) is not valid near the horizon.

The invalidity of (3.11) is important because according to the entanglement wedge reconstruction (more precisely causal wedge reconstruction which is a special case of entanglement wedge reconstruction), $\phi^R(X, \mathcal{O}^{\mathrm{CFT}})$ which satisfies (3.11) should exist. Instead, in the low energy approximation we have

$$\langle 0| \phi^G(X, \mathcal{O}^{\mathrm{CFT}})\phi^G(X', \mathcal{O}^{\mathrm{CFT}}) |0\rangle = \mathrm{tr}_A(\sigma_A \phi^R(X, \mathcal{O}^{\mathrm{CFT}})\phi^R(X', \mathcal{O}^{\mathrm{CFT}})). \qquad (3.12)$$

The subregion duality will also lead to (3.7) and (3.12). Here, we claim that (3.7) and (3.12) can be realized although the causal wedge reconstruction (and thus the entanglement wedge reconstruction) is not correct in the holographic CFT.

This difference between $\phi^G(X, \mathcal{O}^{\mathrm{CFT}})$ and $\phi^R(X, \mathcal{O}^{\mathrm{CFT}})$ is the observer dependence (the patch and foliation dependence). It seems similar to the state dependence of the bulk reconstruction [28] but not the same because we only consider the global vacuum state. Note that the description in the AdS-Rindler patch using $\phi^R(X, \mathcal{O}^{\mathrm{CFT}})$ is not valid near the Rindler horizon because of the finite N effect as we have discussed although the description in the global coordinate using $\phi(X, \mathcal{O}^{\mathrm{CFT}})$ is valid, of course. Nevertheless, the two descriptions are (almost) equivalent in the sense that they reproduce the same correlation functions for the bulk smeared operators supported on $D(M_A)$ except for the near-horizon region.

We call the existence of the two different descriptions depending on the observers, the subregion complementarity. They are different on the entire space in the CFT picture but describe the same bulk physics on the subregion. It might be similar to the black hole complementarity [27]. ϕ^G corresponds to an observer passing through the horizon and ϕ^R does to an external observer.

It is interesting to extend this discussion to more general subregions. We have considered a single spherical subregion (whose size is arbitrary) in the sphere. It is interesting to take A as the union of several spherical subregions. In this case, the bulk correlation functions on the union of the bulk causal wedges of the boundary subregions can be reproduced by the AdS-Rindler type bulk reconstructions for each subregion. The entanglement wedge will play an important role in this analysis although we do not have a concrete one by now. It is important to confirm the subregion complementarity holds for general causal (or entanglement) wedges in the AdS/CFT correspondence as the quantum gravity.

3.2 Using AdS/CFT for the Poincare and Hyperbolic patches

Here, we will provide another derivation of the above statement of the AdS/CFT correspondence for the subregion, i.e. bulk correlators in $D(M_A)$ can be reconstructed from the CFT on subregion $D(A)$, using the Poincare and hyperbolic patches of AdS.

The cylinder metric on subregion $D(A)$ is conformal to that for $\mathbf{R} \times \mathbf{H}^{d-1}$ as

$$-d\tau^2 + d\Omega_{d-1}^2 = e^{2\Phi} \left(-dt_R^2 + dH_{d-1}^2 \right), \tag{3.13}$$

where τ is the global time for cylinder and t_R is the Rindler. The conformal factor $e^{2\Phi}$ is given by (A.10) in Appendix A. Thus, CFT on $D(A)$ is the same as that on $\mathbf{R} \times \mathbf{H}^{d-1}$.

We now focus on $d = 2$. In particular, for $d = 2$, $\mathbf{R} \times \mathbf{H}^{d-1}$ is nothing but the Minkowski space $\mathbf{R}^{1,1}$ with time t_R. If the state is the vacuum for the Rindler Hamiltonian, the dual bulk geometry is the three-dimensional Poincare AdS, and CFT on $D(A)$ is dual to gravity on the Poincare patch.

Now we consider the vacuum state for the global Hamiltonian for time τ. On the subregion A, we consider the reduced density matrix ρ_A, which is a mixed state instead of the Rindler vacuum. More precisely, it is known to be the finite temperature state with $T = 1/(2\pi)$ for the Hamiltonian for time t_R as the usual Unruh effect for this specific choice of the subregion A. Thus, CFT on the subregion A around the global vacuum is the same as CFT on the Minkowski space $\mathbf{R}^{1,1}$ around the finite temperature state.

Forgetting the entire space, we are just considering the holographic CFT on $\mathbf{R}^{1,1}$ at temperature $T = 1/(2\pi)$. The bulk dual should be the Poincare black hole geometry with temperature $T = 1/(2\pi)$ on the Poincare patch. Except for the stretched horizon region where the low-energy approximation breaks down, the (smeared) bulk correlation functions at $T = 1/(2\pi)$ on the outside of the horizon can be reproduced from the CFT on $D(A)$. Furthermore, the black hole solution is known to be equivalent to the AdS-Rindler patch. Thus, the CFT on $D(A)$ around the mixed state ρ_A describes the bulk theory on the AdS-Rindler patch.

The HKLL bulk reconstruction for the black hole solution is the AdS-Rindler bulk reconstruction. Note that this discussion does not imply the bulk local operators reconstructed by the global and Rindler are the same. They are completely different as mentioned in the previous subsection.

The discussion is similar for higher dimensions $d > 2$. If we take the state as the global vacuum, the reduced identity matrix on the spherical subregion A is the finite temperature state with temperature $1/(2\pi)$ for the Hamiltonian for time t_R [21]. Thus, the theory on $D(A)$ is the holographic CFT on $\mathbf{R} \times \mathbf{H}^{d-1}$ with temperature $1/(2\pi)$. The dual geometry should be the $(d + 1)$-dimensional hyperbolic black hole. However, the hyperbolic black hole with temperature $1/(2\pi)$ is the same as the AdS-Rindler geometry (see, e.g., [29] and also Appendix A). Thus the bulk theory on the AdS-Rindler patch can be described by the CFT on $\mathbf{R} \times \mathbf{H}^{d-1}$ around the finite temperature state at $T = 1/(2\pi)$, that is, the CFT on $D(A)$ around the mixed state ρ_A.

We can consider the bulk operator subalgebra $\mathcal{A}_{M_A}$ supported on the bulk subregion M_A, where we assume that the elements of $\mathcal{A}_{M_A}$ are only low-energy operators e.g. by using the projection to the low-energy states. We define the state for algebra $\mathcal{A}_{M_A}$ (in operator algebraic

meaning) by the global vacuum correlation functions. Applying the Gelfand–Naimark–Segal (GNS) construction, we obtain the associated Hilbert space $\mathcal{H}_{M_A}^{\mathrm{GNS}}$. Similarly, we can define $\mathcal{H}_A^{\mathrm{GNS}}$ via the GNS construction from the CFT operator algebra on $\mathbf{R} \times \mathbf{H}^{d-1}$ with the thermal state. The above statement of the matching of the correlation functions means that $\mathcal{H}_{M_A}^{\mathrm{GNS}}$ is isomorphic to $\mathcal{H}_A^{\mathrm{GNS}}$. Under this correspondence, we have a weak version of the JLMS formula, i.e. the equality of the relative entropy

$$S_{\mathrm{bulk}}(\rho_{M_A}|\sigma_{M_A}) = S_{\mathrm{CFT}}(\rho_A|\sigma_A). \tag{3.14}$$

Here ρ_{M_A}, σ_{M_A} are the density matrices in $\mathcal{H}_{M_A}^{\mathrm{GNS}}$ which contain excitation only on M_A, and ρ_A, σ_A are density matrices in $\mathcal{H}_A^{\mathrm{GNS}}$ obtained by the identification between $\mathcal{H}_{M_A}^{\mathrm{GNS}}$ and $\mathcal{H}_A^{\mathrm{GNS}}$. Thus, the equality (3.14) is trivial by construction. The original JLMS formula [8] is a much stronger statement because the original bulk state ρ to define ρ_A can contain the excitation on $M_{\bar{A}}$. Note also that ρ_A in (3.14) is not the reduced density matrix obtained by tracing out $\bar{A}$. For instance, let us consider an excited state in bulk $\phi(X)|0\rangle$ ($X \in M_A$) (more precisely, we need to consider the smeared operator). Then, ρ_A in (3.14) is not the same as $\mathrm{tr}_{\bar{A}}\, \phi^G(X, \mathcal{O}^{\mathrm{CFT}})|0\rangle\langle 0|\, \phi^G(X, \mathcal{O}^{\mathrm{CFT}})$. The point is that $\mathcal{H}_A^{\mathrm{GNS}}$ is the GNS Hilbert space for CFT on the hyperbolic space and not directly related to the entire CFT on the cylinder. By the global HKLL reconstruction, we can consider CFT subalgebra $\mathcal{A}_{M_A}^{\mathrm{CFT}}$ corresponding to bulk subalgebra $\mathcal{A}_{M_A}$. Let $\mathcal{H}_G^{\mathrm{GNS}}$ be the GNS Hilbert space for $\mathcal{A}_{M_A}^{\mathrm{CFT}}$ around the global vacuum. By construction, $\mathcal{H}_G^{\mathrm{GNS}}$ is identified with $\mathcal{H}_{M_A}^{\mathrm{GNS}}$. However, $\mathcal{H}_G^{\mathrm{GNS}}$ and $\mathcal{H}_A^{\mathrm{GNS}}$ are distinct as subalgebras in the full algebra for the entire CFT, though the two spaces can be isomorphic. The existence of the two descriptions $\mathcal{H}_G^{\mathrm{GNS}}$ and $\mathcal{H}_A^{\mathrm{GNS}}$ is the subregion complementarity.

4   Black holes

4.1   Eternal Black holes

It is known that the Rindler space is similar to the black hole by focusing on the near horizon region. The observer with the Rindler time corresponds to the one in the region outside the black hole horizons, i.e. the static observer. In particular, the eternal AdS black hole is similar to the empty AdS case.[12] Indeed, the gravitational theory around the eternal black hole in the asymptotic AdS space-time is believed to be described by the thermofield double state in the tensor product of the identical two CFTs [30]. If we trace out the degrees of freedom in either one of the CFTs, the state becomes the thermal state which corresponds to the region outside the horizon of the eternal black hole. This story is similar to the relation between the global AdS and the AdS-Rindler, where the Schwarzschild-type coordinates outside the horizon correspond to the AdS-Rindler coordinates.

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[12]Indeed, the AdS-Rindler space is a special case of the hyperbolic black holes in AdS as reviewed in Appendix A, and the patch covers the outside region of the horizon.

In the bulk theory around the eternal AdS black hole, we can take a different coordinate system, for example, the Kruskal-type coordinates, in which the whole system is described, as we can take the global coordinates instead of the AdS-Rindler ones in the empty AdS space, although it is not clear that there exists the Hamiltonian corresponding to the Kruskal time in the tensor product of the finite N CFTs.

We expect that the subregion complementarity also holds in the eternal black holes as in the empty AdS case. There are different CFT descriptions of the bulk spacetime outside the horizon depending on the observers (the Schwarzschild-type or the Kruskal-type). For the static observer, which corresponds to either one CFT in the two copies, the region near the horizon cannot be described because the semi-classical approximation breaks down due to the finite N effect as in the AdS-Rinlder case. However, the inside of the horizon can be described by the tensor product of the two CFTs, as in the entire CFT on S^{d-1} for the vacuum case.

4.2   Single-sided black holes

The situation for single-sided black holes is drastically different from the eternal black holes.

Let us consider the black hole in the AdS spacetime that comes from the gravitational collapse, sometimes called the single-sided black hole compared with the eternal black hole. In this case, the dual theory is the single CFT, and the initial state is a pure state corresponding to the bulk object before its collapse. It evolves in time to a typical pure state with the temperature fixed by the energy of the initial state, which corresponds to the black hole in the bulk.[13] This is in the description for the static observer outside the horizon. The semi-classical description is invalid near the horizon as for the AdS-Rindler observer in the empty AdS. This is consistent with the CFT picture, in which the thermal state is in the deconfinement phase of the gauge theory, and there are many ($\mathcal{O}(N^2)$) light modes. These come from the brick wall of the black hole [16] in AdS/CFT [17]. In contrast to the eternal black hole, in this case, the Hilbert space is for the single CFT and then it may be impossible to have another description where the infalling observer will see the smooth spacetime near the horizon.

The absence of an infalling observer implies that there is no semi-classical description at the stretched horizon for single-sided black holes. In this respect, the equivalence principle might be violated for single-sided black holes in quantum gravity ($=$ finite N CFT). This is completely different from eternal black holes where we have semi-classical descriptions corresponding to infalling observers. In eternal black holes, the static observer outside the horizon sees a mixed state. It can be purified by extending the Hilbert space This extended description corresponds to the infalling observer. On the other hand, for the single-sided case, the state is already pure,[14] with no room for extending the Hilbert space. Even if we could extend it, any

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[13]Here, we assume that the black hole is the large AdS-Black hole, which can be in equilibrium with the thermal gas around it by the Hawking radiation.

[14]We consider a finite N CFT on the compact space S^{d-1}. The operator algebra should be type I and there

added degrees of freedom would not entangle with the original ones and thus be completely decoupled. Thus, the black hole complementarity does not hold. As its name suggests, the subregion complementarity holds only when a subregion is under consideration.

The possibility of the violation of the equivalence principle has been proposed by Mathur as the fuzzball conjecture [18, 19] (and later it was also argued as the fire wall [31]).[15] We here expect a brick wall-like behavior near the stretched horizon, where the bulk spacetime picture is not valid because of the shortage of degrees of freedom and the bulk semi-classical theory cannot be applied. Note that this discussion can be applied to a (hypothetical) static heavy star equilibrium with the radiation around it such that the would-be horizon inside the star is very close to the surface of the star. If the distance between them is the order of the Planck length, the shortage of degrees of freedom to reconstruct the bulk theory, like the brick wall model, occurs for the finite N CFT [11] as the black hole case. In this case, the inside of the star cannot be described by the bulk theory.

Acknowledgement

A   General AdS-Rindler and hyperbolic black holes

A.1   AdS-Rindler

$(d + 1)$-dim AdS space is embedded as

$$-(X^{-1})^2 - (X^0)^2 + (X^1)^2 + \cdots + (X^d)^2 = -1. \tag{A.1}$$

The global coordinates (τ, ρ, Ω) are obtained by parameterizing the embedding coordinates as

$$X^{-1} = \frac{1}{\cos \rho} \cos \tau, \quad X^0 = \frac{1}{\cos \rho} \sin \tau, \quad X^i = \tan \rho \, \hat{x}^i(\Omega), \tag{A.2}$$

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exist pure states.

[15]In [32], it was discussed that the fuzzball-like structure should appear in the typical state corresponding to the black hole.

The metric is

$$ds^2 = \frac{1}{\cos^2\rho} \left(-d\tau^2 + d\rho^2 + \sin^2\rho \, d\Omega_{d-1}^2 \right). \tag{A.3}$$

We take the spherical coordinates $\hat{x}^i(\Omega)$ as

$$\hat{x}^1(\Omega) = \cos\theta, \quad \hat{x}^j(\Omega) = \sin\theta \, \hat{y}^j(\mathbf{\Omega}) \quad (j = 2, \ldots, d) \quad \text{with} \quad 0 \le \theta \le \pi, \tag{A.4}$$

where $\hat{y}^j(\Omega)$ $(j = 2, \ldots, d)$ are the embedding of the sphere into $\mathbf{R}^{d-1}$. The asymptotic boundary of the global coordinates is a cylinder $\mathbf{R} \times S^{d-1}$.

The general AdS-Rindler coordinates are given by the following parametrization:

$$X^{-1} = \frac{\sqrt{1 + \xi^2} \cosh\chi + \cos\theta_0 \xi \cosh t_R}{\sin\theta_0}, \quad X^0 = \xi \sinh t_R,$$

$$X^1 = \frac{\xi \cosh t_R + \cos\theta_0 \sqrt{1 + \xi^2} \cosh\chi}{\sin\theta_0}, \quad X^j = \sqrt{1 + \xi^2} \sinh\chi \, \hat{y}^j(\mathbf{\Omega}) \quad (j = 2, \ldots, d). \tag{A.5}$$

The coordinates cover the bulk causal wedge $D(M_A)$ associated with the spherical cap ($0 \le \theta \le \theta_0$) subregion A on the boundary. ($t_R = 0$, $\xi = 0$ surface is the Ryu-Takayanagi surface anchored to the $\theta = \theta_0$ entangling surface at $\tau = 0$ on the boundary.) If $\theta_0 = \pi/2$, the $t_R = 0$ surface covers the half region on the $\tau = 0$ surface (τ is global time).

In these coordinates, the metric becomes

$$ds^2 = -\xi^2 dt_R^2 + \frac{d\xi^2}{1 + \xi^2} + (1 + \xi^2) dH_{d-1}^2, \tag{A.6}$$

where $dH_{d-1}^2 = d\chi^2 + \sinh^2\chi \, d\Omega_{d-2}^2$ is the metric of $(d-1)$-dimensional hyperbolic space $\mathbf{H}^{d-1}$. The asymptotic boundary of the AdS-Rindler coordinates is $\mathbf{R} \times \mathbf{H}^{d-1}$.

On the general AdS-Rindler wedge, we can express the global coordinates $(\tau, \rho, \theta, \mathbf{\Omega})$ in terms of the AdS-Rindler coordinates $(t_R, \xi, \chi, \mathbf{\Omega})$ as

$$\tan\tau = \frac{\sin\theta_0 \xi \sinh t_R}{\sqrt{1 + \xi^2} \cosh\chi + \xi \cos\theta_0 \cosh t_R},$$

$$\cos\rho = \frac{1}{\sqrt{\left(\frac{\sqrt{1+\xi^2} \cosh\chi + \cos\theta_0 \xi \cosh t_R}{\sin\theta_0} \right)^2 + \xi^2 \sinh^2 t_R}},$$

$$\tan\theta = \frac{\sin\theta_0 \sqrt{1 + \xi^2} \sinh\chi}{\xi \cosh t_R + \cos\theta_0 \sqrt{1 + \xi^2} \cosh\chi}. \tag{A.7}$$

On the asymptotic boundary of general AdS-Rindler, (τ, θ) are related with (t_R, χ) as

$$\tan\tau = \frac{\sin\theta_0 \sinh t_R}{\cosh\chi + \cos\theta_0 \cosh t_R}, \quad \tan\theta = \frac{\sin\theta_0 \sinh\chi}{\cosh t_R + \cos\theta_0 \cosh\chi}. \tag{A.8}$$

This coordinate transformation is the same as given in [21]. The cylinder metric $\mathbf{R} \times S^{d-1}$ is conformally mapped to that for $\mathbf{R} \times \mathbf{H}^{d-1}$ as

$$-d\tau^2 + d\Omega_{d-1}^2 = e^{2\Phi}\left(-dt_R^2 + dH_{d-1}^2\right), \tag{A.9}$$

where the conformal factor $e^{2\Phi}$ is given by

$$e^{2\Phi} = \frac{\sin^2\theta_0}{(\cosh\chi + \cos\theta_0 \cosh t_R)^2 + \sin^2\theta_0 \sinh^2 t_R}. \tag{A.10}$$

When $\theta \sim \pi$, that is, the AdS-Rindler wedge covers almost the entire space covered by the Poincare patch, it is more convenient to rescale the coordinates as $t_R = (\sin\theta_0)t$, $\chi = (\sin\theta_0)r$. Then, taking the entire limit $\theta \to \pi$ with only keeping the finite t and r, the relations (A.8) become

$$\tan\tau = \frac{2t}{1 - t^2 + r^2}, \quad \tan\theta = \frac{2x}{1 + t^2 - r^2}. \tag{A.11}$$

This is actually the conformal map between $\mathbf{R} \times S^{d-1}$ and $\mathbf{R} \times \mathbf{R}^{d-1}$.[16] In this limit we have $-dt_R^2 + dH_{d-1}^2 \sim \sin^2\theta_0(-dt^2 + dr^2 + r^2 d\Omega_{d-2}^2)$, and the conformal factor (A.10) becomes $e^{2\Phi} \sim \frac{1}{\sin^2\theta_0}\frac{4}{(1+(t-r)^2)(1+(t+r)^2)}$. $\frac{4}{(1+(t-r)^2)(1+(t+r)^2)}$ is exactly the conformal factor for the map (A.11). Note also that the AdS-Rindler metric (A.6) becomes the Poincare metric

$$\frac{1}{z^2}(-dt^2 + dz^2 + dr^2 + r^2 d\Omega_{d-2}^2), \tag{A.13}$$

by taking the limit $\theta_0 \to \pi$ with the rescaling $t_R = (\sin\theta_0)t$, $\xi = \frac{1}{\sin\theta_0 z}$, $\chi = (\sin\theta_0)r$ and taking only finite t, z, r region.

---

[16]The inverse is

$$t = \frac{\sin\tau}{\cos\tau + \cos\theta}, \quad r = \frac{\sin\theta}{\cos\tau + \cos\theta}. \tag{A.12}$$

A.2 General hyperbolic black holes

The geometry with metric (A.6) represents a hyperbolic black hole with temperature $1/(2\pi)$. The AdS hyperbolic black holes with arbitrary temperature are given by (see, e.g., [29])

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dH_{d-1}^2, \tag{A.14}$$

where

$$f(r) = r^2 - 1 - \frac{\omega^{d-2}}{r^{d-2}}. \tag{A.15}$$

Here, ω is a parameter related to the mass density[17] ϱ of the black hole as

$$\varrho = \frac{d-1}{16\pi G_N}\omega^{d-2}. \tag{A.16}$$

The position of the horizon r_h is determined by $f(r_h) = 0$, i.e.,

$$r_h^{d-2}(r_h^2 - 1) = \omega^{d-2}. \tag{A.17}$$

The temperature of the black hole is given by

$$T = \frac{1}{4\pi}\frac{\partial f}{\partial r}\bigg|_{r=r_h} = \frac{1}{4\pi r_h}(dr_h^2 - d + 2). \tag{A.18}$$

The vanishing mass[18] case ($\omega = 0$) corresponds to the AdS-Rindler metric (A.6) with the coordinate transformation $r^2 = 1 + \xi^2$. It indeed has temperature $T = 1/(2\pi)$.

The d-dimensional holographic CFT on $\mathbf{H}^{d-1}$ at finite temperature should correspond to the gravitational theory around the background (A.14).

B Explicit mirror map for free fields

Here we will explicitly construct the mirror map in a continuum field theory and see the issues. We consider massless free scalar on two-dimensional Minkowski space (T, X). We divide the $T = 0$ time-slice ($-\infty < X < \infty$) into $X < 0$ region and $X > 0$ region, and call their domains of dependence the left and right Rindler wedge, respectively. In each Rindler wedge, we can

---

[17]The ADM mass M is proportional to the (infinite) volume V of the hyperbolic space $\mathbf{H}^{d-1}$. The mass density ϱ means M/V.

[18]The vanishing mass solution is not the lowest mass state. The lowest mass state is given by $\omega^{d-2} = -2d^{-\frac{d}{2}}(d-2)^{\frac{d-2}{2}}$, which means $r_h = \sqrt{\frac{d-2}{d}}$, $\rho = -\frac{(d-1)d^{-\frac{d}{2}}(d-2)^{\frac{d-2}{2}}}{8\pi G_N}$ and $T = 0$.

take the following coordinates:

$$T - X = -e^{-t_R + x_R}, \quad T + X = e^{t_R + x_R} \quad \text{in the right Rindler.} \tag{B.1}$$

$$T - X = e^{t_L - x_L}, \quad T + X = -e^{-t_L - x_L} \quad \text{in the left Rindler.} \tag{B.2}$$

The coordinates (t_R, x_R) and (t_L, x_L) are the Rindler coordinates, and we will sometimes omit the subscripts R, L for simplicity.

In the right wedge, the metric is given by

$$-dT^2 + dX^2 = e^{2x_R}(-dt_R^2 + dx_R^2), \tag{B.3}$$

and the left wedge, it is

$$-dT^2 + dX^2 = e^{-2x_L}(-dt_L^2 + dx_L^2). \tag{B.4}$$

Each metric is conformally flat in the Rindler coordinates, and thus the massless free scalar can be quantized as in Minkowski coordinates. Since we are considering free massless scalar, right-moving and left-moving modes are decoupled, and we here focus on the right-moving field. In the right Rindler wedge, the right-moving field is expanded by the Rindler modes as

$$\phi(x) = \int_0^\infty \frac{dk}{4\pi k} \left(a_k^R e^{-ik(t_R - x_R)} + a_k^{R\dagger} e^{ik(t_R - x_R)} \right). \tag{B.5}$$

The discussion is the same for the left-moving field. $a_k^R, a_k^{R\dagger}$ are the annihilation and creation operators for the right Rindler modes and they satisfy

$$[a_k^R, a_{k'}^{R\dagger}] = 4\pi k \delta(k - k'). \tag{B.6}$$

In the left Rindler wedge, we also have the operators for the left Rindler (right-moving) modes, $a_k^L, a_k^{L\dagger}$, and they also satisfy

$$[a_k^L, a_{k'}^{L\dagger}] = 4\pi k \delta(k - k'). \tag{B.7}$$

Here we consider the thermofield double state for these left and right modes[19]

$$|TFD\rangle = \mathcal{N} \sum_n e^{-\frac{\beta E_n}{2}} |E_n\rangle_L \otimes |E_n\rangle_R, \tag{B.8}$$

where $\mathcal{N}$ is a normalization factor, and $|E_n\rangle_{L/R}$ are Fock states of the Rindler modes with energy E_n for the Rindler Hamiltonian. Note that β should be 2π for the state $|TFD\rangle$ to be

---

[19]We need an IR regulator in order to make the spectrum discrete. Here we consider a problem in UV and thus we do not touch further this point.

the ground state of the Hamiltonian for the Minkowski time T.

We will construct a mirror map m from operators in $\mathcal{A}_L \otimes 1_R$ to ones in $1_L \otimes \mathcal{A}_R$ such that $\mathcal{O}_L \in \mathcal{A}_L \otimes 1_R$ and $\mathsf{m}(\mathcal{O}_L) \in 1_L \otimes \mathcal{A}_R$ act on $|TFD\rangle$ as if they are the same operator, where $\mathcal{A}_{L,R}$ denotes the set of operators on the left or right Fock space, respectively. We can confirm that acting a_k^L (more precisely $a_k^L \otimes 1_R$) to the state $|TFD\rangle$ is the same as acting $e^{-\frac{k}{2}\beta} a_k^{R\dagger}$ as

$$a_k^L |TFD\rangle = e^{-\frac{k}{2}\beta} a_k^{R\dagger} |TFD\rangle. \tag{B.9}$$

We similarly have

$$a_k^{L\dagger} |TFD\rangle = e^{\frac{k}{2}\beta} a_k^R |TFD\rangle. \tag{B.10}$$

Therefore, we have

$$\mathsf{m}(a_k^L) = e^{-\frac{k}{2}\beta} a_k^{R\dagger}, \quad \mathsf{m}(a_k^{L\dagger}) = e^{\frac{k}{2}\beta} a_k^R. \tag{B.11}$$

If we define $a_{-k}^{L,R} = a_k^{L,R\dagger}$, the two equations in (B.11) is unified as $\mathsf{m}(a_k^L) = e^{-\frac{k}{2}\beta} a_{-k}^R$. Note that the map m depends on the choice of the reference state $|TFD\rangle$ as β explicitly appears in (B.11). The mirror of multiple products of ladder operators is also obtained as

$$\mathsf{m}(a_{k_1}^L \cdots a_{k_n}^L) = e^{-\frac{\beta}{2}\sum_i^n k_i} a_{-k_n}^R \cdots a_{-k_1}^R \tag{B.12}$$

as can be understood from the following equations

$$a_{k_1}^L \cdots a_{k_n}^L |TFD\rangle = a_{k_1}^L \cdots a_{k_{n-1}}^L e^{-\frac{k_n}{2}\beta} a_{-k_n}^R |TFD\rangle = e^{-\frac{k_n}{2}\beta} a_{-k_n}^R a_{k_1}^L \cdots a_{k_{n-1}}^L |TFD\rangle, \tag{B.13}$$

where we have used a_k^R commutes with a_k^L.

If the local operator on the left Rindler wedge acts on $|TFD\rangle$, the excited state can also be created *formally* by acting an operator supported on the Right wedge as

$$
\begin{aligned}
\phi_L(t,x) |TFD\rangle &= \int_0^\infty \frac{dk}{4\pi k} \left(a_k^L e^{-ik(t-x)} + a_k^{L\dagger} e^{ik(t-x)} \right) |TFD\rangle \\
&= \int_0^\infty \frac{dk}{4\pi k} \left(a_k^{R\dagger} e^{-ik(t-i\frac{\beta}{2}-x)} + a_k^R e^{ik(t-i\frac{\beta}{2}-x)} \right) |TFD\rangle \\
&= \phi_R(-t+i\beta/2, -x) |TFD\rangle.
\end{aligned}
\tag{B.14}
$$

Here we put subscripts L, R on ϕ to make it obvious that they are operators supported on the left or right wedge. Thus, the mirror operator of $\mathsf{m}(\phi_L(t,x)) = \phi_R(-t+i\beta/2, -x)$. We remark that this is just a formal relation because $\phi_R(-t+i\beta/2, -x)$ is not well-defined due to the blowing up factor $e^{k\beta/2}$ in the momentum representation. We also note that the above mirror map is different from the Tomita-Takesaki modular conjugation.

The position $(t_R, x_R) = (-t + i\beta/2, -x)$ with $\beta = 2\pi$ is $(T - X, T + X) = (e^{t-x}, -e^{-t-x})$ in the Minkowski coordinates which is nothing but the original position that the $\phi_L(t, x)$ is inserted. Thus, the mirror operator $\mathsf{m}(\phi_L(t, x))$ is a singular operator because it forcibly acts as $\phi_L(t, x) \in \mathcal{A}_L$ by an operator in $\mathcal{A}_R$.

In a rigorous sense, the local field $\phi_L(t, x)$ is not well-defined because it contains arbitrary large momentum $k \sim \infty$, and we should smear it. Let us consider the following operator smeared by the Gaussian distribution centering at $t_L = 0, x_L = x_0$ with the width Δ_R (in the Rindler coordinates):

$$
\begin{aligned}
\phi_{\Delta_R}(x_0) &= \frac{1}{\sqrt{2\pi\Delta_R^2}} \int_{-\infty}^{\infty} dx \, e^{-\frac{(x-x_0)^2}{2\Delta_R^2}} \phi_L(t = 0, x) \\
&= \int_0^{\infty} \frac{dk}{4\pi k} \left(e^{-\frac{\Delta_R^2}{2} k^2 + ikx_0} a_k^L + e^{-\frac{\Delta_R^2}{2} k^2 - ikx_0} a_k^{L\dagger} \right),
\end{aligned} \tag{B.15}
$$

where we have used the e.o.m. of the right-moving free scalar: $\phi_L(t, x) = \phi_L(0, x - t)$. The smeared operator effectively consists of the low-momentum modes $k \leq 1/\Delta_R$.

On $|TFD\rangle$, the mirror operator of this smeared operator is given by

$$
\begin{aligned}
\mathsf{m}(\phi_{\Delta_R}) &= \frac{e^{\frac{\beta^2}{8\Delta_R^2}}}{\sqrt{2\pi\Delta_R^2}} \int_{-\infty}^{\infty} dx \, e^{-\frac{(x+x_0)^2 - i\beta(x+x_0)}{2\Delta_R^2}} \phi_R(t = 0, x) \\
&= e^{\frac{\beta^2}{8\Delta_R^2}} \int_0^{\infty} \frac{dk}{4\pi k} \left(e^{-\frac{\Delta_R^2}{2}\left(k - \frac{\beta}{2\Delta_R^2}\right)^2 - ikx_0} a_k^R + e^{-\frac{\Delta_R^2}{2}\left(k + \frac{\beta}{2\Delta_R^2}\right)^2 + ikx_0} a_k^{R\dagger} \right).
\end{aligned} \tag{B.16}
$$

From the first equation, we can find that the mirror operator is centered at $x_R = -x_0$ with the width Δ_R but the smearing function also contains a momentum-shifting factor $e^{\frac{i\beta(x+x_0)}{2\Delta_R^2}}$. From the second equation, we can explicitly see that the dominant momentum is $k \sim \frac{\beta}{2\Delta_R^2}$ for annihilation modes a_k^R, and these modes have exponentially large contributions due to the overall factor $e^{\frac{\beta^2}{8\Delta_R^2}}$. On the other hand, all the creation modes have the suppression factor $e^{-\frac{\Delta_R^2}{2}\left(k + \frac{\beta}{2\Delta_R^2}\right)^2}$ because the integration range is $0 \leq k \leq \infty$. Combining the overall factor $e^{\frac{\beta^2}{8\Delta_R^2}}$, only small momenta $k \sim 0$ has $\mathcal{O}(1)$ contributions to the creation modes.

The dominant momentum $k \sim \frac{\beta}{2\Delta_R^2}$ for annihilation modes is very large for well localized operator $\Delta_R \ll \beta$. This is problematic if we think that our free theory is an effective field theory (EFT) with cutoff $k \lesssim \Lambda$. The original smeared operator (B.15) can be described by the EFT if the smearing width is larger than the cutoff length $\Delta_R \gtrsim 1/\Lambda^{-1}$. However, the mirror operator $\mathsf{m}(\phi_{\Delta_R})$ is problematic if $\Delta_R \ll \beta$ because the dominant momentum can be greater than the cutoff as $k \sim \frac{\beta}{2\Delta_R^2} \gg \Lambda$. Then, the mirror operator description is not valid in the EFT.

The smearing width Δ_R which we have considered is the coordinate width, and the physical width is $e^{-x_0}\Delta_R$ for $\Delta_R \ll 1$. Thus, if we considered a smeared field localized well in the region $X_0 - \frac{\Delta_M}{2} \le X \le X_0 + \frac{\Delta_M}{2}$ in the Minkowski coordinate, we should take $x_0 = -\log(-X_0)$ and $\Delta_R = e^{x_0}\Delta_M = \frac{\Delta_M}{(-X_0)}$ where we suppose X_0 is negative to consider the operator supported on the left wedge. It means that if the center of the smeared operator is near the Rindler horizon $X_0 \sim 0$, Δ_R can be $\mathcal{O}(1)$ even if $\Delta_M \ll 1$. Thus, in this case, the dominant momentum $k \sim \frac{\beta}{2\Delta_R^2}$ does not become large and the above problem of the mirror operator may be avoided. Nevertheless, far from the horizon, the mirror map is still problematic.

We finally remark that the mirror map differs from the holographic subregion map considered in section 3, i.e., the map from ϕ^G to ϕ^R in CFT. The mirror map does not preserve the Hermiticity of operators as can be understood from (B.11), and does not preserve the order of the multiplicity as (B.12), while the holographic subregion map preserves them.

C On AdS-Rindler mode expansion

Here, we concentrate on the $d = 2$ case for simplicity. For the free scalar field theory in the three-dimensional AdS space, the bulk local operator $\phi(\tau, \rho, \theta)$ in the global patch can be expanded by the modes in the global AdS as

$$\phi(\tau, \rho, \theta) = \sum_{n,m} \left(a_{nm}^{\text{global}\,\dagger} e^{i\omega_{nm}\tau} e^{-im\theta} + a_{nm}^{\text{global}} e^{-i\omega_{nm}\tau} e^{im\theta} \right) \psi_{nm}^{\text{bulk}}(\rho) \tag{C.1}$$

where n is a non-negative integer, m is an integer, $\omega_{nm} = 2n + |m| + \Delta$ and $\psi_{nm}^{\text{bulk}}(\rho)$ are modes in ρ-direction. The explicit form is given by $\psi_{nm}^{\text{bulk}}(\rho) = \frac{1}{\mathcal{N}_{nm}} \sin^{|m|}(\rho) \cos^\Delta(\rho) P_n^{|m|,\Delta-1}(\cos(2\rho))$, where $\mathcal{N}_{nm}$ is the numerical constant given in [2] and $P_n^{|m|,\Delta-1}$ is the Jacobi polynomial.

We now take the half subregion (i.e. take $\theta_0 = \pi/2$ in Appendix A). In the right AdS-Rindler wedge, the mode expansion of the bulk local operator is given by

$$\phi(t_R, \xi, \chi) = \int_0^\infty d\omega \int_{-\infty}^\infty d\lambda \frac{1}{\sqrt{2\pi}} \tilde{\psi}_{\omega,\lambda}(\xi) \left[a_{\omega,\lambda} e^{-i\omega t_R + i\lambda\chi} + a_{\omega,\lambda}^\dagger e^{i\omega t_R - i\lambda\chi} \right] \tag{C.2}$$

where

$$\tilde{\psi}_{\omega,\lambda}(\xi) = \frac{N_{\omega,\lambda}}{\Gamma(\nu+1)} \xi^{i\omega} (1+\xi^2)^{-\frac{i\omega}{2}-\frac{\Delta}{2}} \,_2F_1\left(\frac{i\omega - i\lambda + \nu + 1}{2}, \frac{i\omega + i\lambda + \nu + 1}{2}; \nu+1; \frac{1}{1+\xi^2} \right), \tag{C.3}$$

with $\nu = \Delta - \frac{d}{2} = \Delta - 1$ and

$$N_{\omega,\lambda} = \frac{\left| \Gamma\left(\frac{i\omega - i\lambda + \nu + 1}{2} \right) \right| \left| \Gamma\left(\frac{i\omega + i\lambda + \nu + 1}{2} \right) \right|}{\sqrt{4\pi\omega} |\Gamma(i\omega)|}. \tag{C.4}$$

The coordinate transformation of the global and AdS-Rindler coordinates are explicitly given by

$$\cos\rho = \frac{1}{\sqrt{1+\xi^2}\cosh\chi}, \quad \tan\theta = \frac{\sqrt{1+\xi^2}}{\xi}\sinh\chi, \tag{C.5}$$

on the $\tau = 0$ ($t_R = 0$) slice.

Let us consider the following smeared bulk local operator on $t_R = 0, \xi = \xi_0, \chi = \chi_0$ in the AdS-Rindler wedge:

$$\phi^\Lambda(\xi_0, \chi_0) = \frac{1}{2\pi\Lambda\Lambda'}\int_0^\infty d\xi \int_{-\infty}^\infty d\chi\, e^{-\frac{\Lambda^2}{2}(\xi-\xi_0)^2 - \frac{\Lambda'^2}{2}(\chi-\chi_0)^2}\phi(t_R = 0, \xi, \chi). \tag{C.6}$$

Here, $\Lambda, \Lambda' \gg 1$, $\Lambda' = \mathcal{O}(\Lambda)$ and $\Lambda\xi_0 \gg 1$.[20] We will expand it by the global modes as $\phi^\Lambda(\xi_0, \chi_0) = \sum_{n,m}\beta^{nm}a_{n,m}^{\text{global}} + h.c.$ and evaluate the coefficient

$$\beta^{nm} = \frac{1}{2\pi\Lambda\Lambda'}\int_0^\infty d\xi \int_{-\infty}^\infty d\chi\, e^{-\frac{\Lambda^2}{2}(\xi-\xi_0)^2 - \frac{\Lambda'^2}{2}(\chi-\chi_0)^2}e^{im\theta}\psi_{nm}^{\text{bulk}}(\rho). \tag{C.7}$$

In order to do so, we need the large n, m behavior of $\psi_{nm}^{\text{bulk}}(\rho)$. The bulk e.o.m. is approximately given by

$$\left((\partial_\rho)^2 + (2n + |m|)^2 - \frac{1}{\sin^2\rho}m^2\right)\psi_{nm}^{\text{bulk}}(\rho) \simeq 0, \tag{C.8}$$

and the solution around $\rho = \rho_0$ is $\psi_{nm}^{\text{bulk}}(\rho_0 + \delta\rho) \simeq C_+ e^{in\alpha\delta\rho} + C_- e^{-in\alpha\delta\rho}$ with

$$\alpha^2 = 4(1+a) - \frac{a^2}{\tan^2\rho_0}, \tag{C.9}$$

where $C_\pm$ are n-independent constants and $a \equiv |m|/n$. Here, $\rho_0 = \rho(\xi_0, \chi_0)$ by the coordinate transformation (C.5). Thus, we have $\psi_{nm}^{\text{bulk}}(\rho_0 + \delta\rho) \sim \cos(n\alpha\delta\rho + C)$ for $\alpha^2 > 0$ or exponential for $\alpha^2 < 0$. This is consistent with the asymptotic behavior of the Jacobi polynomial given in [33, 34], where $\psi_{nm}^{\text{bulk}}(\rho) \sim n^{C(\rho,a)}\cos(A(\rho,a)n + B(\rho,a))$ for $\alpha^2 > 0$, which can be written as $a < 2\frac{\sin\rho_0}{\cos^2\rho_0}(1 + \sin\rho_0)$, and $\psi_{nm}^{\text{bulk}}(\rho) \sim n^{\tilde{C}(\rho,a)}e^{-\tilde{A}(\rho,a)n + \tilde{B}(\rho,a)}$ for $\alpha^2 < 0$.

Then, β^{nm} given in (C.7) for large n, m is exponentially suppressed with the factor $e^{-\tilde{A}(\rho_0,a)n}$ for $\alpha^2 < 0$. For $\alpha^2 > 0$, by the Gaussian integration in (C.7) it has the Gaussian damping factor $e^{-\frac{n^2}{2\Lambda^2}(c_1)^2 - \frac{n^2}{2\Lambda'^2}(c_2)^2}$, where $c_1 = \left.\left(\alpha\frac{\partial\rho}{\partial\xi} + a\frac{\partial\theta}{\partial\xi}\right)\right|_{\xi=\xi_0,\chi=\chi_0}$ and $c_2 = \left.\left(\alpha\frac{\partial\rho}{\partial\chi} + a\frac{\partial\theta}{\partial\chi}\right)\right|_{\xi=\xi_0,\chi=\chi_0}$ for $n \gg \Lambda$. Thus, for generic ξ_0, χ_0, the smeared bulk local operator $\phi^\Lambda(\xi_0, \chi_0)$ in the AdS-Rindler

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[20] With the last condition, the Gaussian factor is small on $\xi = 0$. Instead of ξ, we can consider the Gaussian smearing for, for example, $\xi' = \ln(e^\xi - 1)$, which takes $-\infty < \xi' < \infty$. Such a modification is not important for the qualitative properties of $\phi^\Lambda(\xi_0, \chi_0)$ and the results below are unchanged.

patch contains mostly low energy modes of $a_{n,m}^{\text{global}}$. Here, low energy means $n, |m| = \mathcal{O}(\Lambda)$.

However, if the center (ξ_0, χ_0) of the smearing is near the AdS-Rindler horizon $\xi = 0$, the smeared bulk local operator contains the high energy modes as we will see below. Near $\xi = \xi_0, \chi = \chi_0$ with $\xi_0 \simeq 0$, the metric becomes approximately

$$ds^2 \simeq \xi_0^2(-t_R^2 + d\zeta^2 + d\tilde{\chi}^2), \tag{C.10}$$

where

$$\zeta = \frac{\xi - \xi_0}{\xi_0}, \quad \tilde{\chi} = \frac{\chi - \chi_0}{\xi_0}. \tag{C.11}$$

Under this coordinate system, the cut-offs for the momenta of $\zeta, \tilde{\chi}$ will be identified as the cut-off of the energy that conjugates to the Rindler time t_R because the metric is the one of the Minkowski spacetime. We will denote these as Λ^R, Λ'^R, which are given by $\Lambda^R = \xi_0 \Lambda$, $\Lambda'^R = \xi_0 \Lambda'$ from (C.6). In the global coordinates, where the Rindler horizon is at $\theta = \pm\pi/2$, the coordinate transformation is

$$\delta\rho \simeq \frac{1}{\cosh\chi_0}\xi_0\tilde{\chi}, \quad \delta\theta \simeq \frac{1}{\sinh\chi_0}\xi_0\zeta. \tag{C.12}$$

Using the large n, m approximation, $e^{i\theta m}\psi_{nm}^{\text{bulk}}(\rho) \sim e^{im\delta\theta}\cos(n\alpha\delta\rho + C)$, we find that β^{nm} is exponentially small if $n \gg \frac{1}{\xi_0}\Lambda'^R$ or $|m| \gg \frac{1}{\xi_0}\Lambda^R$.[21]

Thus, the smeared bulk local operator at $\xi = \xi_0$ in the AdS-Rindler patch with the energy cut-off $\mathcal{O}(\Lambda^R)$ contains the modes in the global coordinate with the energy of $\mathcal{O}(\frac{1}{\xi_0}\Lambda^R)$. Because the finite N CFT has many but finite degrees of freedom determined by N, there is an upper bound for the energy of the modes which is expected to be the Planck energy Λ_{Planck}. If $\xi_0 \leq 1/\Lambda_{\text{Planck}}$, because $\Lambda^R \gg 1$, the smeared bulk local operator in the AdS-Rindler patch consists of the trans-Planckian energy modes of global coordinates and then it cannot be well-defined. Note that the surface $\xi_0 = 1/\Lambda_{\text{Planck}}$ is the stretched horizon, which is away from the horizon by the Planck length because the metric becomes $ds^2 \simeq -\xi_0 t_R^2 + d\xi^2 + d\chi^2$ near the horizon. This is regarded as the brick wall argument [16, 17] without the artificial Dirichlet boundary condition for the AdS-Rindler case.

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[21] The asymptotic boundary $\xi \to \infty$ is also special. Near $\xi = \xi_0, \chi = \chi_0$ with $\xi_0 \simeq \infty$, the metric becomes approximately $ds^2 \simeq \xi_0^2(-t_R^2 + d\zeta^2 + d\chi^2)$ where $\zeta = -\frac{\xi - \xi_0}{(\xi_0)^2}$. The cut-off for the momenta of ζ is given by $\Lambda^R = (\xi_0)^2\Lambda$. With $\delta\rho \simeq -\frac{1}{\sqrt{1+\cosh^2\chi}}\frac{1}{(\xi_0)^2}\delta\xi$ and $\tan\theta \simeq \sinh\chi$, we find that β^{nm} is exponentially small if $n \gg \Lambda^R$ or $|m| \gg \Lambda'$. This is the same as for a generic ξ_0, χ_0.

D   Bulk coherent states

Here we consider a free massive scalar field in $(d+1)$-dimensional AdS space (global patch). Let $f_{nlm}(x)$ be the positive frequency solutions of the EoM,

$$f_{nlm}(x) = e^{-i\omega_{nl}\tau}\psi_{nl}(\rho)Y_{lm}(\Omega) \tag{D.1}$$

with

$$\omega_{nl} = 2n + l + \Delta, \quad \Delta = \frac{d}{2} + \sqrt{m^2 + \frac{d^2}{4}} = \frac{d}{2} + \nu. \tag{D.2}$$

$\psi_{nl}(\rho)$ represents the modes for ρ-direction whose explicit form is not important here. We introduce the Klein-Gordon inner product in the global AdS as

$$(f_1, f_2) := i \int dV (f_1^*\partial_\tau f_2 - (\partial_\tau f_1^*)f_2), \tag{D.3}$$

where $\int dV = \int_0^{\frac{\pi}{2}} d\rho \int d\Omega_{d-1}(\tan\rho)^{d-1}$. Using the inner product, we normalize the modes as $(f_{nlm}, f_{n'l'm'}) = \delta_{nn'}\delta_{ll'}\delta_{mm'}$.

We can expand the scalar field by these modes,

$$\phi(x) = \sum_{n,l,m} \left(a_{nlm}f_{nlm}(x) + a_{nlm}^\dagger f_{nlm}^*(x) \right). \tag{D.4}$$

In our normalization, we have the commutation relation

$$[a_{nlm}, a_{n'l'm'}^\dagger] = \delta_{nn'}\delta_{ll'}\delta_{mm'}. \tag{D.5}$$

Let us consider an arbitrary classical configuration $\phi_{cl}(x)$ that satisfies the classical EoM. This solution can be expanded by the modes f_{nlm} as

$$\phi_{cl}(x) = \sum_{n,l,m} (B_{nlm}f_{nlm}(x) + B_{nlm}^*f_{nlm}^*(x)) \tag{D.6}$$

with the coefficients B_{nlm} given by the Klein-Gordon inner product

$$B_{nlm} = (f_{nlm}, \phi_{cl}). \tag{D.7}$$

We then introduce the following anti-Hermitian operator

$$B = \sum_{n,l,m} \left(-B_{nlm}^*a_{nlm} + B_{nlm}a_{nlm}^\dagger \right). \tag{D.8}$$

It satisfies

$$[\phi(x), B] = \phi_{cl}(x). \tag{D.9}$$

Thus, the state

$$|\Psi\rangle = e^B |0\rangle \tag{D.10}$$

is the coherent state corresponding to the classical configuration $\phi_{cl}(x)$ as

$$\langle\Psi| \phi(x) |\Psi\rangle = \phi_{cl}(x) \tag{D.11}$$

where we have used that the vacuum 1-pt function $\langle 0| \phi |0\rangle$ vanishes.

Note also that the operator B can be written using the Klein-Gordon inner product as

$$B = -(\phi_{cl}, \phi)_{\tau=0}. \tag{D.12}$$

By decomposing the integration region in the inner product, i.e. $\tau = 0$ slice, into M_A and $M_{\bar{A}}$, B is decomposed as $B = B_{M_A} + B_{M_{\bar{A}}}$ where $B_{M_A}, B_{M_{\bar{A}}}$ are operators supported only in $M_A, M_{\bar{A}}$ respectively. Then, the state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = e^{B_{M_A}} e^{B_{M_{\bar{A}}}} |0\rangle \tag{D.13}$$

because B_{M_A} and $B_{M_{\bar{A}}}$ commute with each other. Note that $e^{B_{M_A}}, e^{B_{M_{\bar{A}}}}$ are unitary.

Since $|\Psi\rangle$ is a coherent state, we have

$$\langle\Psi| \phi(x)\phi(x') |\Psi\rangle - \langle 0| \phi(x)\phi(x') |0\rangle = \phi_{cl}(x)\phi_{cl}(x'). \tag{D.14}$$

The change in the expectation value of the energy-momentum tensor is

$$\langle\Psi| T_{MN}^{\text{bulk}} |\Psi\rangle - \langle 0| T_{MN}^{\text{bulk}} |0\rangle = T_{MN}^{\text{bulk}}[\phi_{cl}], \tag{D.15}$$

where $T_{MN}^{\text{bulk}}[\phi_{cl}]$ is the energy-momentum tensor for the classical configuration ϕ_{cl}. The right-hand side of (D.15) is UV-finite.

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