

# Geometry of General Hypersurfaces in Spacetime: Junction Conditions.

Marc Mars and José M. M. Senovilla

Departament de Física Fonamental, Universitat de Barcelona  
Diagonal 647, 08028 Barcelona, Spain

and

Laboratori de Física Matemàtica, Societat Catalana de Física, IEC.

## Abstract

We study imbedded *general* hypersurfaces in spacetime i.e. hypersurfaces whose timelike, spacelike or null character can change from point to point. *Inherited* geometrical structures on these hypersurfaces are defined by two distinct methods: the first one, in which a *rigging vector* (a vector not tangent to the hypersurface anywhere) induces the standard *rigged connection*; and the other one, more adapted to physical aspects, where each observer in spacetime induces a completely new type of connection that we call the *rigged metric connection* which is volume preserving. The generalisation of the Gauss and Codazzi equations are also given. With the above machinery, we attack the problem of matching two spacetimes across a general hypersurface. It is seen that the *preliminary junction conditions* allowing for the correct definition of Einstein's equations in the distributional sense reduce to the requirement that the first fundamental form of the hypersurface be continuous, because then, there exists a maximal  $C^1$  atlas in which the metric is continuous. The Bianchi identities are then proven to hold in the distributional sense. Next, we find the *proper junction conditions* which forbid the appearance of singular parts in the curvature. These are shown equivalent to the existence of coordinate systems where the metric is  $C^1$ . Finally, we derive the physical implications of the junction conditions: only six independent discontinuities of the Riemann tensor are allowed. These are six matter discontinuities at non-null points of the hypersurface. For null points, the existence of two arbitrary discontinuities of the Weyl tensor (together with four in the matter tensor) are also allowed. The classical results for timelike, spacelike or null hypersurfaces are trivially recovered.

PACS numbers: 04.20.Cv, 02.40.+m.

# 1 Introduction.

The purpose of this paper is twofold. First, we wish to find the possible geometrical structures that *general* hypersurfaces inherit from the spacetime manifold. And second, we study the application of these results to the matching of spacetimes across one of those general hypersurfaces.

Here, *general* hypersurface means an imbedded three-dimensional manifold without specifying anything about its timelike, spacelike or null character, which will also be permitted to *change from point to point*. In principle, it might seem that general hypersurfaces in the above sense are not very physical, given that physical particles travel along timelike lines which will never become null, least of all spacelike. But this is a very simplified naive view, and in fact general hypersurfaces are commonplace in General Relativity. A few examples are: Gödel's universe, where there are no imbedded hypersurfaces without boundary which are spacelike everywhere [1]; the apparent horizon of Vaidya's radiating metric [2] (whenever the mass function becomes a constant after a while); or even something as simple as the stationary limit surface of Kerr's vacuum solution [1], which is everywhere timelike except at points on the axis where it is null and tangent to the horizon. The reader can also find his/her own preferred examples.

The trouble with general hypersurfaces is that they inherit metrics from the spacetime (the first fundamental form) which can be, at some points, degenerate, and also the signature is not constant. Therefore, they do not have an intrinsic inherited Riemannian structure, and the usual metric connection cannot be defined in them. The degeneracy of the first fundamental form appears also in null hypersurfaces, but for general hypersurfaces we have the change of signature as an added problem. Imagine, for example, a hypersurface which is spacelike in some open set, then changes to a null hypersurface. In the spacelike part, we might be tempted to define the canonical Riemannian structure defined by the non-degenerate first fundamental form, but then this structure (the connection) blows up as we approach the region where the hypersurface is null. For null hypersurfaces we do not have this problem, as the metric is degenerate everywhere. Thus, we see that, even in the case of a general hypersurface with just one single point of different signature, we cannot define the affine structure which would be good enough for the hypersurface without that point.

The study of degenerate metrics has been addressed since long ago, but usually keeping the signature (or the degeneracy) constant. The question is simply how to define a good connection in general hypersurfaces which is somehow induced by the affine structure of the whole manifold. The first references we are aware of are those of Bortolotti [3],[4], where he studied absolutely specialized degenerate metrics, or in more simple terms, metrics with non-zero eigenvectors with zero eigenvalue and vanishing Lie derivative along these vectors. A little later, Hlavatý [5] defined canonical and unique induced connections for general hypersurfaces in which the second fundamental

form is non-degenerate. This is also explained in the famous Schouten book [6], which is an unavoidable reference for all these matters, and where the rigged connection we define in Sect.3 is thoroughly studied. This type of rigged connection has also been used later several times with different purposes (see, for example, [7],[8],[9],[10],[11],[12],[13]). The paper by Bonnor [13] is specially interesting in the sense that there appears the possible importance of using observers to define the rigged connections for the first time. We shall elaborate on this idea in Sect.4, where we provide a completely new way of inducing a connection onto general hypersurfaces. We call this new connection the rigged metric connection, and we claim that it is physically reasonable. By means of this connection, we shall also be able to define an induced volume element in the hypersurface which is preserved by the rigged metric connection, something which was not possible with the old rigged connections in general.

The second part of this paper deals with the junction conditions which must be imposed when the matching hypersurface is a general one, where we apply the results developed in the first part. As far as we know, this problem has not been considered previously. The junction conditions for time- or space-like hypersurfaces are known since the work of Darmois [14], Lichnerowicz [15] and O'Brien-Synge [17] and the relations between them were established by Bonnor & Vickers [18] (see also Bel & Hamoui [19]). More recently, the junction conditions for null hypersurfaces have also been studied by several authors [20], [21], and a unified treatment of the three types of hypersurfaces has been produced as well in [22],[23]. However, the problem of generalizing the junction conditions to arbitrary hypersurfaces, which we carry out in Sect.5, is of most interest in Gravitation. Consider, for example, the question of phase transitions in the early universe, which can take place at spacelike regions, the information being transmitted from there on by causal (null) signals. The resulting whole hypersurface of discontinuous change is thus formed by a spacelike region and the boundary of its causal future, which is null. Yet another example. Imagine that we wish to match the Kerr vacuum solution to some interior spacetime and that the matching hypersurface turns out to be, *precisely*, the stationary limit surface. In all these and other similar cases, we need to know the proper junction conditions for general hypersurfaces.

Finally, in Sect.6 we complete our study by deducing the physical implications of the junction conditions, that is, the allowable discontinuities in the matter contents and in the pure gravitational field once a proper matching has been done. The continuity of the normal components of the energy-momentum tensor, usually called Israel's conditions, were known for the case of non-null hypersurfaces since the work by Israel [24]. The generalization of these conditions to the case of null hypersurfaces appears in the recent paper by Barrabs and Israel [22]. Here, we present in the last section the generalization of these conditions to arbitrary matching hypersurfaces. We also give the continuity properties of the general Riemann tensor and of the Weyl tensor, the latter representing the pure gravitational field. To our knowledge, these conditions were previously unknown, despite of its evident interest for problems involving shock

gravitational waves, or even for traditional cases of timelike or spacelike matching hypersurfaces. In this last case, we prove that the possible Weyl discontinuities are forced by those in the matter contents, the latter being the truly independent allowed discontinuities, while at null points of a general hypersurface arbitrary discontinuities not related to the matter contents are possible in two of the ten independent components of the Weyl tensor.

## 2 General Definitions and Basic Results.

We will consider throughout this paper an oriented four-dimensional Riemannian manifold  $(V_4, g)$ , with signature  $+2$ , and a hypersurface defined in it. More strictly, let us also have an orientable three-dimensional manifold  $\Sigma$ , and a  $C^3$  map  $\Phi$  from this manifold to  $V_4$

$$\begin{aligned}\Phi : \Sigma &\rightarrow V_4 \\ \xi &\rightarrow \Phi(\xi) \equiv x(\xi),\end{aligned}$$

which is an imbedding [1] of  $\Sigma$  into  $V_4$ . Locally, the hypersurface  $\Phi(\Sigma) \in V_4$  can be defined by a function  $F$  from  $V_4$  to the real numbers,  $\mathbb{R}$ , through the equation  $F(x) = 0$ .

As usual, we can construct the pull-back of covariant tensors in  $V_4$  and the push-forward of contravariant tensors in  $\Sigma$ . For any point  $p \in \Sigma$ , the imbedding  $\Phi$  allows us to write the differential map (or push-forward) from the tangent plane of  $\Sigma$  at  $p$ ,  $T_p(\Sigma)$ , to the tangent plane of  $V_4$  at  $\Phi(p)$ ,  $T_{\Phi(p)}(V_4)$ ,

$$\begin{aligned}d\Phi|_p : T_p(\Sigma) &\rightarrow T_{\Phi(p)}(V_4) \\ \vec{V} &\rightarrow d\Phi|_p(\vec{V}),\end{aligned}$$

which is of rank 3 at any  $p \in \Sigma$ , and its generalization to contravariant tensors of any order in  $\Sigma$ . Similarly, the pull-back maps the dual tangent plane of  $V_4$  at  $\Phi(p)$ ,  $T_{\Phi(p)}^*(V_4)$ , onto the dual tangent plane of  $\Sigma$  at  $p$ ,  $T_p^*(\Sigma)$ ,

$$\begin{aligned}\Phi^*|_p : T_{\Phi(p)}^*(V_4) &\rightarrow T_p^*(\Sigma) \\ \omega &\rightarrow \Phi^*|_p(\omega),\end{aligned}$$

and it is again of maximum rank. This map can also be extended to covariant tensors of any order in the manifold  $V_4$ .

Because of the rank of the push-forward, we have that  $d\Phi|_p(T_p(\Sigma))$  is a three-dimensional linear subspace of  $T_{\Phi(p)}(V_4)$  and it is the tangent plane to the hypersurface in  $\Phi(p)$ . We will denote this tangent plane at  $\Phi(p)$  as  $T_p\Sigma$ . Let us take a coordinate system on  $\Sigma$ ,  $\{\xi^a\}$  where  $a$  runs from 1 to 3, in a neighbourhood of a point  $p \in \Sigma$ ,

and a coordinate system on  $V_4$ ,  $\{x^\alpha\}$  where  $\alpha$  goes from 0 to 3, in a neighbourhood of  $\Phi(p) \in V_4$ . The three vectors  $\left.\frac{\partial}{\partial \xi^a}\right|_p$  constitute a basis for  $T_p(\Sigma)$  and the push-forward maps them into three linearly independent vectors at  $\Phi(p)$  which are a basis for the tangent plane to the hypersurface. Therefore we can write

$$d\Phi|_p \left( \left.\frac{\partial}{\partial \xi^a}\right|_p \right) = \left.\frac{\partial \Phi^\mu}{\partial \xi^a} \frac{\partial}{\partial x^\mu}\right|_{\Phi(p)} \equiv e_a^\mu \left.\frac{\partial}{\partial x^\mu}\right|_{\Phi(p)} \equiv \vec{e}_a|_{\Phi(p)}.$$

As vector fields,  $\vec{e}_a$  are defined only on the hypersurface  $\Phi(\Sigma)$ . Given that the map  $\Phi$  is an homeomorphism between  $\Sigma$  and  $\Phi(\Sigma)$ , we will from now on identify the points  $p$  and  $\Phi(p)$  and the sets  $\Sigma$  and  $\Phi(\Sigma)$  in order to simplify the notation. We can consider the orthogonal complement of the tangent plane  $T_p\Sigma$  in the dual space  $T_p^*(V_4)$ , which is obviously a one-dimensional linear subspace. This one-dimensional subspace is generated by a non-zero one-form at  $p$  that we will denote by  $\mathbf{n}|_p$ , which is uniquely defined up to a non-zero multiplicative factor  $\sigma(p)$ , and is called the normal form of the hypersurface or simply the normal to the hypersurface. We define the normal vector to the hypersurface as the vector obtained by raising the index of  $\mathbf{n}$  with the metric of  $V_4$ . As a consequence of its definition,  $\mathbf{n}$  is only defined on  $\Sigma$ . In the coordinate basis  $\mathbf{n} = n_\mu dx^\mu$  and we have  $\mathbf{n}(\vec{e}_a)|_p = 0$  or, in components,  $n_\mu e_a^\mu|_p = 0$ .

The fact that  $V_4$  is a Riemannian manifold with metric tensor  $g$  allows us to define uniquely a symmetric two-covariant tensor in  $\Sigma$  by using the pull-back. This symmetric tensor  $\Phi^*(g)$  will be called  $\bar{g}$  and is the *first fundamental form* of the hypersurface. In the basis  $\{d\xi^a\}$  the components of  $\bar{g}$  are  $\bar{g}_{ab} = g_{\alpha\beta} e_a^\alpha e_b^\beta$  and of course it is defined only on the hypersurface. There is also another two-covariant tensor  $K$  on  $\Sigma$  defined as  $K = \Phi^*(\nabla\mathbf{n})$ , where  $\mathbf{n}$  is any extension of the one-form field  $\mathbf{n}$  outside the hypersurface. The definition of  $K$  is independent of this extension, as is evident from the expression of its components in a coordinate system

$$K_{ab} = e_a^\mu e_b^\nu \nabla_\mu n_\nu.$$

This tensor on  $\Sigma$  is obviously symmetric and is called the *second fundamental form* of the hypersurface.

Using the canonical volume form  $\boldsymbol{\eta}$  of  $V_4$  one can find an explicit expression of the normal  $\mathbf{n}$  in terms of the basis vectors of the tangent plane to the hypersurface. In components we have

$$n_\mu = A^{-1} \eta_{\mu\alpha\beta\gamma} e_1^\alpha e_2^\beta e_3^\gamma, \quad (1)$$

where  $A$  is an arbitrary scalar function on  $\Sigma$ , different from zero everywhere, which reflects the freedom that exists in choosing the normal to the hypersurface. A simple calculation shows that the norm of the normal form is

$$n_\mu n^\mu = -A^{-2} \det({}^3\bar{g}), \quad (2)$$

where  ${}^3\bar{g}$  is the determinant of the first fundamental form on the hypersurface. So we can write the following very well known result.

**Lemma 1** *At a point  $p \in \Sigma$  the first fundamental form is degenerate if and only if the normal vector is null at  $p$ .*

We define the volume element 3-form on  $\Sigma$ ,  $\eta_{abc}$ , by

$$\eta_{123} n_\alpha \equiv \eta_{\alpha\beta\gamma\delta} e_1^\beta e_2^\gamma e_3^\delta \quad (3)$$

or, equivalently,  $\eta_{abc} = A\delta_{abc}$  where  $\delta_{abc}$  is the standard alternating symbol of Levi-Civita. Last expression shows that this volume element depends on the normalization factor of  $n_\alpha$  and we see that fixing by any means the volume element on the hypersurface is equivalent to choosing the normalization factor of the normal form. We define also the contravariant volume element on  $\Sigma$  as  $\eta^{abc} \equiv A^{-1}\delta^{abc}$  in order to satisfy the usual property in Riemannian manifolds:  $\eta_{abc}\eta^{def} = \delta_{abc}^{def}$  where  $\delta_{abc}^{def}$  is the Kronecker tensor.

Let  $p$  be any point in the hypersurface, the tangent vectors to the hypersurface at  $p$  can be uniquely characterized as the vectors  $\vec{V} \in T_p\Sigma$  such that  $\mathbf{n}|_p(\vec{V}) = 0$ . Therefore  $\vec{n} \cdot \vec{n}|_p = 0$  is equivalent to  $\vec{n}|_p \in T_p\Sigma$ , where  $\vec{n}$  is the normal vector to the hypersurface and the dot means scalar product with the metric in  $V_4$ . For a point  $p$  in the hypersurface we can consider the set of vectors in  $T_p(V_4)$  that are orthogonal to the tangent plane  $T_p\Sigma$ . So we define

$$\perp T_p\Sigma \equiv \left\{ \vec{V} \in T_p(V_4); \quad g(\vec{V}, \vec{Y}) = 0 \quad \forall \vec{Y} \in T_p\Sigma \right\} = \langle \vec{n}|_p \rangle \quad .$$

This set is obviously a one-dimensional linear subspace of  $T_p(V_4)$  and it is generated by  $\vec{n}|_p$ . We have already seen that these two linear subspaces,  $T_p\Sigma$  and  $\perp T_p\Sigma$ , will have non-zero vectors in common if and only if the normal vector at  $p$  is null. So we can write the following

**Lemma 2**  $\langle \vec{n}|_p \rangle \cap T_p\Sigma = \{0\} \Leftrightarrow \vec{n} \cdot \vec{n}|_p \neq 0 \Leftrightarrow T_p(V_4) = \langle \vec{n}|_p \rangle \oplus T_p\Sigma$ .

Here the second equivalence follows immediately from the first one because  $\langle \vec{n}|_p \rangle$  is one-dimensional and  $T_p\Sigma$  is three-dimensional.

Let us now briefly recall the usual case of hypersurfaces whose normal vector is not null at any point, i.e.  $\vec{n} \cdot \vec{n} \neq 0$  everywhere on  $\Sigma$  and by continuity the sign of  $\vec{n} \cdot \vec{n}$  must be constant on the whole hypersurface. By Lemma 1 we know that the first fundamental form on the hypersurface is not degenerate and then  $\Sigma$  is a Riemannian manifold that, in consequence, possesses a unique connection associated with the metric. We will find explicitly this connection in a way that can be easily generalised to the case of general hypersurfaces. We have at any point  $p \in \Sigma$  the decomposition of the tangent plane  $T_p(V_4) = \langle \vec{n}|_p \rangle \oplus T_p\Sigma$  and then any vector  $\vec{V} \in T_p(V_4)$  can be decomposed uniquely into its parallel and its orthogonal part  $\vec{V} = \vec{V}_\perp + \vec{V}_\parallel$  where the parallel

component  $\vec{V}_{\parallel} \in T_p\Sigma$  and the orthogonal component  $\vec{V}_{\perp} \in \langle \vec{n}|_p \rangle$ . As a consequence of standard results in the theory of dual spaces we can decompose the dual tangent plane as  $T_p^*(V_4) = \langle \mathbf{n}|_p \rangle \oplus A_p^{\perp}$ , where  $\langle \mathbf{n}|_p \rangle$  is the linear space orthogonal to  $T_p\Sigma$  (in the sense of dual spaces, not of the metric) which is obviously generated by the normal one-form  $\mathbf{n}$ , and  $A_p^{\perp}$  is the 3-dimensional linear subspace orthogonal to  $\langle \vec{n}|_p \rangle$  defined as  $A_p^{\perp} \equiv \{ \boldsymbol{\omega} \in T_p^*(V_4); \boldsymbol{\omega}(\vec{n}|_p) = 0 \}$ . With this decomposition of the tangent plane we can define a map  $T$  from the whole tangent plane  $T_p(V_4)$  onto the tangent plane to the hypersurface  $T_p\Sigma$  by assigning to any vector in the tangent plane its component parallel to the hypersurface

$$\begin{aligned} T : T_p(V_4) &\rightarrow T_p\Sigma \\ \vec{A} &\rightarrow \vec{A}_{\parallel} \end{aligned}$$

This map is linear and has rank 3 at any point  $p \in \Sigma$ . Considering the definitions of the pull-back and the normal form  $\mathbf{n}$  we have that  $\text{Ker}(\Phi^*) = \langle \mathbf{n}|_p \rangle$ . The decomposition of the dual tangent plane at the point  $p$  and the fact that the rank of the pull-back is 3 allows us to establish that  $\Phi^*$  is an isomorphism between  $A_p^{\perp}$  and  $T_p^*\Sigma$ . Therefore there exists an inverse map, that we will call  $\Lambda$ , from  $T_p\Sigma$  onto  $A_p^{\perp}$  which assigns to any one-form on the hypersurface,  $\boldsymbol{\Omega} \in T_p^*\Sigma$ , the unique one-form on the manifold,  $\Lambda(\boldsymbol{\Omega}) \in T_p^*(V_4)$  with the properties  $\Phi^*(\Lambda(\boldsymbol{\Omega})) = \boldsymbol{\Omega}$  and  $[\Lambda(\boldsymbol{\Omega})](\vec{n})|_p = 0$ . These two maps,  $T$  and  $\Lambda$ , can be respectively generalised to act on contravariant and covariant tensors of any order.

From now on and for the sake of simplicity in the notation, we will use the same symbol to denote a vector (or vector field) tangent to the hypersurface considered as a vector in the manifold  $V_4$  or as a vector in the three-dimensional manifold  $\Sigma$ . Let us then consider two vector fields  $\vec{X}$  and  $\vec{Y}$  defined on the hypersurface and tangent to it everywhere, that is to say:  $\forall p \in \Sigma, \vec{X}|_p, \vec{Y}|_p \in T_p\Sigma$ . The vector field  $\nabla_{\vec{X}}\vec{Y}$  is well defined on the hypersurface in the sense that there is no need of extending  $\vec{X}$  or  $\vec{Y}$  out of the hypersurface in order to calculate it. However,  $\nabla_{\vec{X}}\vec{Y}$  can have, in general, a non-zero orthogonal component. Discarding this orthogonal component we obtain the operation  $\bar{\nabla}_{\vec{X}}\vec{Y}$  defined as  $\bar{\nabla}_{\vec{X}}\vec{Y} \equiv T(\nabla_{\vec{X}}\vec{Y}) \equiv (\nabla_{\vec{X}}\vec{Y})_{\parallel}$  which is a covariant derivative without torsion on the hypersurface. The standard proof of this result can be found in [1],[6], and it is important to note that this proof makes use nowhere of the fact that the vector field  $\vec{n}$  is the normal to the hypersurface. However, this is the key point in proving the second property of this connection, namely:  $\bar{\nabla}$  is the unique metric connection associated with the metric  $\bar{g}$  of the hypersurface.

As a final remark regarding this metric connection, let us mention that a very simple calculation shows [1],[6] that its Riemann tensor,  $R_{bcd}^a$ , verifies the two following well-known relations called *the Gauss equation*:

$$e_{d\mu} R_{\alpha\beta\gamma}^{\mu} e_a^{\alpha} e_b^{\beta} e_c^{\gamma} = R_{abc}^f \bar{g}_{fd} - \frac{1}{\vec{n} \cdot \vec{n}} K_{bd} K_{ca} + \frac{1}{\vec{n} \cdot \vec{n}} K_{cd} K_{ba}$$

which is obviously independent on the normalization of  $\mathbf{n}$ , and *the Codazzi equation*:

$$n_\mu R^\mu_{\alpha\beta\gamma} e_a^\alpha e_b^\beta e_c^\gamma = \bar{\nabla}_c K_{ba} - \bar{\nabla}_b K_{ca} - \frac{1}{2(\vec{n} \cdot \vec{n})} K_{ba} \bar{\nabla}_c (\vec{n} \cdot \vec{n}) + \frac{1}{2(\vec{n} \cdot \vec{n})} K_{ca} \bar{\nabla}_b (\vec{n} \cdot \vec{n}).$$

With the usual normalization  $\vec{n} \cdot \vec{n} = \pm 1$ , the last two terms of this equation vanish and the Codazzi equation takes the standard, more simplified, form.

Let us now return to the case of general hypersurfaces and try to generalise the previous construction. The main fact that has allowed us to define a covariant derivative on the hypersurface was the decomposition of the tangent plane  $T_p(V_4)$  at any point  $p \in \Sigma$ . In the general case, however, it is not true that the normal vector  $\vec{n}$  does not belong to the tangent plane  $T_p\Sigma$  for every  $p$ , so we cannot follow exactly the same steps as before. To avoid this difficulty, let us define a *rigged* hypersurface [6] as a hypersurface  $\Sigma$  where we have taken a vector field which does not belong to the tangent plane  $T_p\Sigma$  anywhere. This vector field,  $\vec{\ell}$ , called the *rigging*, is defined only on  $\Sigma$  and, obviously, it can be chosen in many different ways. The question now is to find out the structure that the riggings induce on  $\Sigma$  and then try to fix one (or some) of them with specially desirable properties.

Given a rigging  $\vec{\ell}$ , we can decompose the tangent plane at every point  $p \in \Sigma$  as  $T_p(V_4) = \langle \vec{\ell}|_p \rangle \oplus T_p\Sigma$  and therefore, analogously as before, the dual tangent plane at  $p$  is decomposed as  $T_p^*(V_4) = \langle \mathbf{n}|_p \rangle \oplus A_p^{\vec{\ell}}$ , where  $A_p^{\vec{\ell}}$  is the dual orthogonal to  $\langle \vec{\ell}|_p \rangle$ . It is evident that  $A_p^{\vec{\ell}}$  is a three-dimensional linear subspace of  $T_p^*(V_4)$  which depends on the particular choice of the rigging  $\vec{\ell}$ . As before, we can define the linear maps  $T$  and  $\Lambda$  and its generalizations to tensors of any order. The decomposition written above does not change if we multiply the rigging by a factor depending on the point of the hypersurface and, due to the fact that  $\mathbf{n}(\vec{\ell})$  must be different from zero, we can always choose this factor such that  $\mathbf{n}(\vec{\ell}) = 1$  everywhere on  $\Sigma$ . Then, the vector fields  $\{\vec{\ell}, \vec{e}_a\}$  constitute a basis of the tangent planes to  $V_4$  at any point on  $\Sigma$  and the dual basis is given by  $\{\mathbf{n}, \omega^a\}$  satisfying

$$\ell^\alpha \omega_\alpha^a = 0, \quad \omega_\alpha^a e_b^\alpha = \delta_b^a, \quad n_\alpha e_a^\alpha = 0, \quad n_\alpha \ell^\alpha = 1.$$

The pull-back and push-forward and the maps  $T$  and  $\Lambda$  can be made explicit when considered in that basis as follows. First of all, let  $\Xi$  be an arbitrary covariant tensor field in  $V_4$  whose components in the coordinate basis  $\{dx^\alpha\}$  are  $\Xi_{\alpha_1 \dots \alpha_q}$ . The pull-back of this tensor is a covariant tensor on the hypersurface with components in the basis  $\{d\xi^a\}$

$$[\Phi^*(\Xi)]_{a_1 \dots a_q} = \Xi_{\gamma_1 \dots \gamma_q} e_{a_1}^{\gamma_1} e_{a_2}^{\gamma_2} \dots e_{a_q}^{\gamma_q}.$$

Similarly, for an arbitrary contravariant tensor on the hypersurface,  $\Upsilon$ , with components  $\Upsilon^{a_1 \dots a_r}$  in the basis  $\left\{ \frac{\partial}{\partial \xi^a} \right\}$ , the push-forward gives a contravariant tensor in  $V_4$



with components in the coordinate basis  $\left\{\frac{\partial}{\partial x^\alpha}\right\}$

$$[d\Phi(\Upsilon)]^{\gamma_1 \dots \gamma_r} = \Upsilon^{a_1 \dots a_r} e_{a_1}^{\gamma_1} e_{a_2}^{\gamma_2} \dots e_{a_r}^{\gamma_r} .$$

The map  $T$  assigns to any contravariant tensor on  $V_4$ , say  $\Theta$ , with components  $\Theta^{\gamma_1 \dots \gamma_r}$  in the basis  $\left\{\frac{\partial}{\partial x^\alpha}\right\}$  a contravariant tensor on the hypersurface which, in the basis  $\left\{\frac{\partial}{\partial \xi^a}\right\}$ , has the following components

$$[T(\Theta)]^{a_1 \dots a_r} = \Theta^{\gamma_1 \dots \gamma_r} \omega_{\gamma_1}^{a_1} \omega_{\gamma_2}^{a_2} \dots \omega_{\gamma_r}^{a_r} .$$

Finally, for an arbitrary covariant tensor in the hypersurface,  $\Delta$ , whose components in the basis  $\{d\xi^a\}$  are  $\Delta_{a_1 \dots a_q}$ , the map  $\Lambda$  produces a covariant tensor in the manifold  $V_4$  with components in the basis  $\{dx^\alpha\}$

$$[\Lambda(\Delta)]_{\gamma_1 \dots \gamma_q} = \Delta_{a_1 \dots a_q} \omega_{\gamma_1}^{a_1} \omega_{\gamma_2}^{a_2} \dots \omega_{\gamma_q}^{a_q} .$$

From these four expressions we observe the intrinsic definition of the pull-back and push-forward, and the dependence of  $T$  and  $\Lambda$  on the rigging  $\vec{\ell}$ . Some particular cases of these relations concerning the rigging and normal vectors and that we will use later in this paper are

$$\begin{aligned} [T(\vec{n})]^a &\equiv n^a = n^\alpha \omega_\alpha^a , & [\Phi^*(\ell)]_a &\equiv \ell_a = \ell_\alpha e_a^\alpha , \\ [\Phi^*(\mathbf{n})]_a &\equiv n_a = n_\alpha e_a^\alpha = 0 , & [T(\vec{\ell})]^a &\equiv \ell^a = \ell^\alpha \omega_\alpha^a = 0 , \\ n^a \ell_a &= 1 - (\vec{n} \cdot \vec{n}) (\vec{\ell} \cdot \vec{\ell}) , & \bar{g}_{ab} n^b &= -(\vec{n} \cdot \vec{n}) \ell_a . \end{aligned} \quad (4)$$

Given an arbitrary tensor field in  $V_4$ , defined at least on the hypersurface, we can define another tensor field in  $V_4$ , defined *only* on  $\Sigma$ , by transporting it first into the hypersurface and then back towards the manifold. If the tensor, say  $W$ , has components  $W_{\gamma_1 \dots \gamma_q}^{\beta_1 \dots \beta_r}$ , the image tensor, denoted  $\tilde{W}$ , will have components

$$\tilde{W}_{\gamma_1 \dots \gamma_q}^{\beta_1 \dots \beta_r} = W_{\rho_1 \dots \rho_q}^{\delta_1 \dots \delta_r} e_{a_1}^{\rho_1} \omega_{\gamma_1}^{a_1} e_{a_2}^{\rho_2} \omega_{\gamma_2}^{a_2} \dots e_{a_q}^{\rho_q} \omega_{\gamma_q}^{a_q} e_{b_1}^{\beta_1} \omega_{\delta_1}^{b_1} e_{b_2}^{\beta_2} \omega_{\delta_2}^{b_2} \dots e_{b_r}^{\beta_r} \omega_{\delta_r}^{b_r}$$

as can be easily checked. The object  $P_\beta^\gamma \equiv e_a^\gamma \omega_\beta^a$  appears here in a natural way. Using the decomposition of the unit tensor  $\delta_\beta^\gamma = \ell^\gamma n_\beta + e_1^\gamma \omega_\beta^1 + e_2^\gamma \omega_\beta^2 + e_3^\gamma \omega_\beta^3$  we find the explicit expression

$$P_\beta^\gamma \equiv e_a^\gamma \omega_\beta^a = \delta_\beta^\gamma - n_\beta \ell^\gamma . \quad (5)$$

The following properties show that  $P_\beta^\gamma$  is the projection tensor to the hypersurface (with respect to the rigging)

$$P_\beta^\gamma e_a^\beta = e_a^\gamma , \quad P_\beta^\gamma \ell^\beta = 0 , \quad P_\beta^\gamma n_\gamma = 0 , \quad P_\beta^\gamma \omega_\gamma^a = \omega_\beta^a , \quad P_\beta^\gamma P_\delta^\beta = P_\delta^\gamma , \quad P_\gamma^\gamma = 3 .$$

Thus,  $\tilde{W}$  is the complete projection to the hypersurface of  $W$  (with respect to the rigging) in the sense that

$$\ell^{\gamma_i} \tilde{W}_{\gamma_1 \dots \gamma_q}^{\beta_1 \dots \beta_r} = 0 \quad \forall i = 1 \dots q , \quad n_{\beta_j} \tilde{W}_{\gamma_1 \dots \gamma_q}^{\beta_1 \dots \beta_r} = 0 \quad \forall j = 1 \dots r .$$

### 3 First Connection in a General Hypersurface: The Rigged Connection.

In this section we generalize, to the case of general hypersurfaces, the results seen in the previous section for non-null hypersurfaces. Let us consider then three vector fields  $\vec{X}$ ,  $\vec{Y}$  and  $\vec{Z}$  on  $\Sigma$ , which are tangent everywhere to the hypersurface, and let us construct the operator  $\bar{\nabla}_{\vec{X}}\vec{Y} \equiv T(\nabla_{\vec{X}}\vec{Y}) \equiv (\nabla_{\vec{X}}\vec{Y})_{\parallel}$  where now the parallel part is taken with respect to the decomposition of the tangent plane defined by the rigging vector  $\vec{\ell}$ . We have again the following result [6]:

**Theorem 1** *For each rigging, the operation  $\bar{\nabla}_{\vec{X}}\vec{Y}$  is a torsion-free covariant derivative on  $\Sigma$ .*

The proof follows exactly the same steps than that of non-null hypersurfaces mentioned above. Nevertheless, in this case we cannot prove that this connection is metric with respect to the first fundamental form, as is obvious because we do not have in general a canonical non-degenerate metric on the hypersurface.

We call this connection the *rigged connection*. Let us find now an explicit expression for its Christoffel symbols in the coordinate basis  $\vec{e}_a$ . First of all we note that if we decompose the covariant derivative  $\nabla_{\vec{X}}\vec{Y}$  into its parallel part and  $\vec{\ell}$ -part and use the definition of the second fundamental form we find

$$\nabla_{\vec{X}}\vec{Y} = (\nabla_{\vec{X}}\vec{Y})_{\parallel} - K(\vec{X}, \vec{Y})\vec{\ell} . \quad (6)$$

But the Christoffel symbols are defined by  $\bar{\nabla}_{\vec{e}_a}\vec{e}_b = \Gamma_{ba}^c\vec{e}_c$  and then from the definition of the covariant derivative  $\bar{\nabla}$  and the above expression we immediately get  $\Gamma_{ba}^c\vec{e}_c = \nabla_{\vec{e}_a}\vec{e}_b + K(\vec{e}_a, \vec{e}_b)\vec{\ell}$ , and contracting now with  $\omega^c$  we obtain

$$\Gamma_{ba}^c = \omega^c e_a^\nu \nabla_\nu e_b^\mu , \quad \Gamma_{ab}^c = \Gamma_{ba}^c . \quad (7)$$

We will now relate the Riemann tensors of  $V_4$  and of the hypersurface in order to generalize the Gauss and Codazzi equations. Let us first recall that the definition of the curvature tensor of any connection is [1]

$$R(\vec{X}, \vec{Y})\vec{Z} = \nabla_{\vec{X}}\nabla_{\vec{Y}}\vec{Z} - \nabla_{\vec{Y}}\nabla_{\vec{X}}\vec{Z} - \nabla_{[\vec{X}, \vec{Y}]}\vec{Z} , \quad (8)$$

but using repeatedly equation (6) we have

$$\begin{aligned} \nabla_{\vec{X}}\nabla_{\vec{Y}}\vec{Z} &= \nabla_{\vec{X}}(\bar{\nabla}_{\vec{Y}}\vec{Z} - K(\vec{Y}, \vec{Z})\vec{\ell}) = \bar{\nabla}_{\vec{X}}\bar{\nabla}_{\vec{Y}}\vec{Z} - K(\vec{X}, \bar{\nabla}_{\vec{Y}}\vec{Z})\vec{\ell} - \nabla_{\vec{X}}(K(\vec{Y}, \vec{Z})\vec{\ell}) , \\ \nabla_{[\vec{X}, \vec{Y}]}\vec{Z} &= \bar{\nabla}_{[\vec{X}, \vec{Y}]}\vec{Z} - K([\vec{X}, \vec{Y}], \vec{Z})\vec{\ell} \end{aligned}$$

so that putting all this into formula (8) we find

$$R(\vec{X}, \vec{Y}) \vec{Z} = \overline{R}(\vec{X}, \vec{Y}) \vec{Z} - K(\vec{X}, \overline{\nabla}_{\vec{Y}} \vec{Z}) \vec{\ell} + K(\vec{Y}, \overline{\nabla}_{\vec{X}} \vec{Z}) \vec{\ell} - \nabla_{\vec{X}}(K(\vec{Y}, \vec{Z}) \vec{\ell}) + \nabla_{\vec{Y}}(K(\vec{X}, \vec{Z}) \vec{\ell}) + K([\vec{X}, \vec{Y}], \vec{Z}) \vec{\ell}. \quad (9)$$

Contracting this expression with an arbitrary 1-form  $\alpha \in A^{\vec{\ell}}$ , so that  $\alpha(\vec{\ell}) = 0$ , we find the desired generalization of Gauss equation

$$\alpha(R(\vec{X}, \vec{Y}) \vec{Z}) = \alpha(\overline{R}(\vec{X}, \vec{Y}) \vec{Z}) - K(\vec{Y}, \vec{Z}) \alpha(\nabla_{\vec{X}} \vec{\ell}) + K(\vec{X}, \vec{Z}) \alpha(\nabla_{\vec{Y}} \vec{\ell}) \quad (10)$$

where there appears, in a natural way, a new 1-contravariant, 1-covariant tensor on  $\Sigma$  defined by  $\Psi(\alpha, \vec{X}) \equiv \alpha(\nabla_{\vec{X}} \vec{\ell})$  or, in components [6]

$$\Psi_b^a = \omega_\mu^a e_b^\nu \nabla_\nu \ell^\mu \quad (11)$$

and thus, Gauss' equation takes the form

$$\omega_\alpha^d R_{\beta\gamma\delta}^\alpha e_a^\beta e_b^\gamma e_c^\delta = \overline{R}_{abc}^d - K_{ac} \Psi_b^d + K_{ab} \Psi_c^d. \quad (12)$$

Analogously, contracting (9) with the normal form  $\mathbf{n}$  we get

$$\mathbf{n}(R(\vec{X}, \vec{Y}) \vec{Z}) = K(\vec{Y}, \overline{\nabla}_{\vec{X}} \vec{Z}) - K(\vec{X}, \overline{\nabla}_{\vec{Y}} \vec{Z}) + K([\vec{X}, \vec{Y}], \vec{Z}) + \nabla_{\vec{Y}}(K(\vec{X}, \vec{Z})) - \nabla_{\vec{X}}(K(\vec{Y}, \vec{Z})) - K(\vec{Y}, \vec{Z}) \mathbf{n}(\nabla_{\vec{X}} \vec{\ell}) + K(\vec{X}, \vec{Z}) \mathbf{n}(\nabla_{\vec{Y}} \vec{\ell})$$

which can be simplified to the following expression, that we call *Codazzi-1 equation*,

$$n_\mu R_{\alpha\beta\gamma}^\mu e_a^\alpha e_b^\beta e_c^\gamma = \overline{\nabla}_c K_{ba} - \overline{\nabla}_b K_{ca} + K_{ba} \varphi_c - K_{ca} \varphi_b \quad (13)$$

where again a one-form in  $\Sigma$  arises naturally:  $\varphi(\vec{X}) \equiv \mathbf{n}(\nabla_{\vec{X}} \vec{\ell})$ . Its components are

$$\varphi_a = n_\mu e_a^\nu \nabla_\nu \ell^\mu. \quad (14)$$

From the Ricci identity  $(\nabla_{\vec{e}_a} \nabla_{\vec{e}_b} - \nabla_{\vec{e}_b} \nabla_{\vec{e}_a}) \ell^\mu = R_{\alpha\beta\gamma}^\mu \ell^\alpha e_a^\beta e_b^\gamma$  and contracting first with the three forms  $\omega^c$  and second with the normal form  $\mathbf{n}$  one can easily find the following equations

$$\omega_\mu^c R_{\alpha\beta\gamma}^\mu \ell^\alpha e_a^\beta e_b^\gamma = \overline{\nabla}_a \Psi_b^c - \overline{\nabla}_b \Psi_a^c + \varphi_b \Psi_a^c - \varphi_a \Psi_b^c, \quad (15)$$

$$n_\mu R_{\alpha\beta\gamma}^\mu \ell^\alpha e_a^\beta e_b^\gamma = \overline{\nabla}_a \varphi_b - \overline{\nabla}_b \varphi_a + K_{cb} \Psi_a^c - K_{ca} \Psi_b^c \quad (16)$$

which we call *Codazzi-2 and Codazzi-3 equations*, respectively.

We shall also write down another equation involving the one-form  $\ell$  which is obtained from the Ricci identity applied to that one-form and contracting later with the tangent vectors  $\vec{e}_c$ . The equation reads

$$\ell_\mu R_{\alpha\beta\gamma}^\mu e_a^\alpha e_b^\beta e_c^\gamma = \overline{\nabla}_c \mathcal{H}_{ab} - \overline{\nabla}_b \mathcal{H}_{ac} + \frac{1}{2} \partial_b (\vec{\ell} \cdot \vec{\ell}) K_{ac} - \frac{1}{2} \partial_c (\vec{\ell} \cdot \vec{\ell}) K_{ab} \quad (17)$$

and obviously it is not independent of Codazzi-1 and Gauss equations. There appears, however, a new tensor in the hypersurface, in general not symmetric, defined as

$$\mathcal{H}_{ab} = e_a^\mu e_b^\nu \nabla_\mu \ell_\nu \quad (18)$$

which will play a central role in the discussion of the junction conditions in Sect.5. Codazzi's equations (13,15) and expression (17) collapse into the unique standard Codazzi equation in the case of non-null hypersurfaces everywhere when the rigging is chosen in the usual way as the normal vector to the hypersurface. The third Codazzi equation vanishes identically in that case.

With the definitions we already have in  $\Sigma$ , we can easily derive the following formulas for the directional derivatives of different objects along the vectors  $\vec{e}_a$ :

$$\nabla_{\vec{e}_a} \vec{e}_b = -K_{ab} \vec{\ell} + \Gamma_{ab}^c \vec{e}_c \quad , \quad (19)$$

$$\nabla_{\vec{e}_a} \mathbf{n} = -\varphi_a \mathbf{n} + K_{ab} \boldsymbol{\omega}^b \quad , \quad (20)$$

$$\nabla_{\vec{e}_a} \vec{\ell} = \varphi_a \vec{\ell} + \Psi_a^b \vec{e}_b \quad , \quad (21)$$

$$\nabla_{\vec{e}_a} \boldsymbol{\omega}^b = -\Psi_a^b \mathbf{n} - \Gamma_{ac}^b \boldsymbol{\omega}^c \quad . \quad (22)$$

We established in the last section a way to relate tensors in the manifold  $V_4$  with tensors in  $\Sigma$ . Now, by means of the previous equations, we are going to establish a general relation between the covariant derivatives in the manifold and in the hypersurface. To that aim, let  $S_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_r}$  be a tensor field in  $V_4$  defined at least on every point of  $\Sigma$ . By means of the pull-back and the map  $T$ , one can assign to this tensor another tensor field, defined in the hypersurface, with components

$$S_{b_1 \dots b_q}^{a_1 \dots a_r} = \omega_{\mu_1}^{a_1} \dots \omega_{\mu_r}^{a_r} e_{b_1}^{\nu_1} \dots e_{b_q}^{\nu_q} S_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_r} \quad . \quad (23)$$

Then, it is not difficult to prove that the projection to the hypersurface of the covariant derivative of  $S_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_r}$  is related to the covariant derivative of  $S_{b_1 \dots b_q}^{a_1 \dots a_r}$  in the following form

$$\begin{aligned} \omega_{\mu_1}^{a_1} \dots \omega_{\mu_r}^{a_r} e_{b_1}^{\nu_1} \dots e_{b_q}^{\nu_q} e_c^\gamma \nabla_\gamma S_{\nu_1 \dots \nu_q}^{\mu_1 \dots \mu_r} &= \bar{\nabla}_c S_{b_1 \dots b_q}^{a_1 \dots a_r} + \sum_{i=1}^r S_{b_1 \dots b_q}^{a_1 \dots a_{i-1} \gamma a_{i+1} \dots a_r} n_\gamma \Psi_c^{a_i} + \\ &+ \sum_{j=1}^q S_{b_1 \dots b_{j-1} \gamma b_{j+1} \dots b_q}^{a_1 \dots a_r} \ell^\gamma K_{cb_j} \quad , \end{aligned} \quad (24)$$

where the mixed tensor  $S_{b_1 \dots b_q}^{a_1 \dots a_{i-1} \gamma a_{i+1} \dots a_r}$  is the projection of the original tensor in the surface on all its indexes except  $\gamma$  and analogously for  $S_{b_1 \dots b_{j-1} \gamma b_{j+1} \dots b_q}^{a_1 \dots a_r}$ . Immediate consequences of this equation are

$$\bar{\nabla}_c \bar{g}_{ab} + \ell_b K_{ca} + \ell_a K_{cb} = 0 \quad , \quad (25)$$

$$\bar{\nabla}_c \ell_a = -(\vec{\ell} \cdot \vec{\ell}) K_{ca} + \bar{g}_{ab} \Psi_c^b + \ell_a \varphi_c = -(\vec{\ell} \cdot \vec{\ell}) K_{ca} + \mathcal{H}_{ca} \quad . \quad (26)$$

From the Riemann tensor of the rigged connection we can define, following the usual notation of the Ricci calculus [6], the tensor fields

$$\bar{R}_{bd} = \bar{R}_{bad}^a \quad (\text{Ricci tensor}), \quad (27)$$

$$V_{cd} = \bar{R}_{acd}^a . \quad (28)$$

This last tensor is not identically zero because the connection in the hypersurface is not metric in general. Keeping in mind that the torsion of the connection vanishes, these tensors just defined and the Riemann tensor satisfy the following identities

1.  $\bar{R}_{bcd}^a = -\bar{R}_{bdc}^a$  .
2.  $\bar{R}_{bcd}^a + \bar{R}_{cdb}^a + \bar{R}_{dbc}^a = 0$  (First Bianchi identity).
3.  $V_{cd} = -V_{dc}$  .
4.  $V_{cd} = \bar{R}_{cd} - \bar{R}_{dc}$  .
5.  $\bar{\nabla}_e \bar{R}_{bcd}^a + \bar{\nabla}_c \bar{R}_{bde}^a + \bar{\nabla}_d \bar{R}_{bec}^a = 0$  (Second Bianchi identity).
6.  $\bar{\nabla}_a V_{bc} + \bar{\nabla}_b V_{ca} + \bar{\nabla}_c V_{ab} = 0 \Leftrightarrow d\mathbf{V} = 0$  .

Last identity shows that the two-form  $\mathbf{V}$  is closed and then, because of the Poincaré Lemma, it is locally exact. Actually it is globally exact as the following result proves:

**Proposition 1** *The two-form  $\mathbf{V}$  is, in fact, exact and it is related with  $\varphi$  defined above by  $\mathbf{V} = -d\varphi$  .*

*Proof:* The proof we present here of this result makes a main use of the Gauss and Codazzi-3 equations. Contracting the first two indexes in the Gauss equation, we find  $V_{bc} = \bar{R}_{abc}^a = K_{ac}\Psi_b^a - K_{ab}\Psi_c^a + R_{\alpha\beta\gamma}^\mu e_a^\alpha e_b^\beta e_c^\gamma \omega_\mu^a$  and using now formula (5) we obtain  $V_{bc} = K_{ac}\Psi_b^a - K_{ab}\Psi_c^a - n_\mu R_{\alpha\beta\gamma}^\mu \ell^\alpha e_b^\beta e_c^\gamma$ . Codazzi-3 equation leads us then to

$$V_{bc} = -(\partial_b \varphi_c - \partial_c \varphi_b) ,$$

where we have used that the rigged connection is torsion-free, and the proof is complete.

On the other hand, the expression of the Riemann tensor in terms of the Christoffel symbols  $\mathbf{V}$  in terms of them:

$$V_{bc} = \partial_b \Gamma_{ac}^a - \partial_c \Gamma_{ab}^a ,$$

and this together with the previous expression involving  $V_{bc}$  suggests the definition of the geometric object

$$\Gamma_c \equiv \varphi_c + \Gamma_{ac}^a$$

which verifies  $\partial_b \Gamma_c - \partial_c \Gamma_b = 0$ . We cannot say that  $\Gamma_c$  is a closed 1-form because  $\Gamma_c$  is not a tensor. However, in each coordinate system of the hypersurface there is a function  $\phi$  such that, locally

$$\Gamma_c = \partial_c \phi .$$

Until now we have not considered the transformation of the above defined objects by choosing the rigging in another direction. Let us change the direction of the rigging without changing the normalization of the normal form and maintaining the condition  $n_\mu \ell^\mu = 1$  everywhere. Under these assumptions, the most general change of the rigging field is

$$\ell'^\mu = \ell^\mu + s^\mu \tag{29}$$

where the vector  $s^\mu$  verifies  $s^\mu n_\mu = 0$  and therefore  $s^\mu = s^a e_a^\mu$ . This change of the rigging induces the following transformation on the objects depending on it

$$\begin{aligned} \omega'^a &= \omega^a - s^a \mathbf{n}, \\ \Gamma'_{bc} &= \Gamma_{bc} + s^a K_{bc}, \\ \varphi'_a &= \varphi_a - s^b K_{ba}, \\ \Psi'_b &= \Psi_b - s^a \varphi_b + s^a s^c K_{bc} + \bar{\nabla}_b s^a \end{aligned} \tag{30}$$

and from these expressions we immediately obtain the following interesting result.

**Proposition 2** *The object  $\Gamma_c$  does not depend on the rigging.*

In consequence, the function  $\phi$  related with  $\Gamma_c$  as was explained above does not depend on the direction of the rigging. Let us find now how the object  $\Gamma_c$  transforms under a change of coordinates  $\xi^a = \xi'^a (\xi'^a)$  in  $\Sigma$ . If we call  $A_{a'}^a = \frac{\partial \xi^a}{\partial \xi'^a}$  the jacobian matrix of this transformation, the transformation law for  $\Gamma_c$  is

$$\Gamma_{a'} = A_{a'}^a \Gamma_a + \frac{1}{\det(A)} \partial_{a'} \det(A)$$

Using now that in both coordinate systems  $\Gamma_c$  is the partial derivative of a function  $\phi$ , we can easily relate the function in a coordinate system with the function in the other one by  $\phi' = \phi + \log(|\det(A)|)$ , or equivalently,  $e^{\phi'} = |\det(A)| e^\phi$ . Thus, for orientation preserving coordinate changes,  $e^\phi$  transforms as a scalar density of weight +1 and therefore, it may have something to do with the volume element in the hypersurface. In fact, this relation is concreted as follows.

**Proposition 3** *Except for a multiplicative constant, the function  $e^\phi$  is the unique independent component of the volume form already defined in the hypersurface.*

*Proof:* From definition (3) and using the normalization condition  $\ell^\mu n_\mu = 1$  we have  $\eta_{123} = \ell^\alpha e_1^\beta e_2^\gamma e_3^\delta \eta_{\alpha\beta\gamma\delta}$ , expression that, despite of its appearance, does not depend on the specific choice of  $\vec{\ell}$ . If we calculate now

$$\partial_a (\eta_{123}) = e_a^\rho \nabla_\rho (\ell^\alpha e_1^\beta e_2^\gamma e_3^\delta \eta_{\alpha\beta\gamma\delta}) = \eta_{\alpha\beta\gamma\delta} e_a^\rho \nabla_\rho (\ell^\alpha e_1^\beta e_2^\gamma e_3^\delta)$$

and make use of formulas (19), (21) and the complete antisymmetry of  $\eta_{\alpha\beta\gamma\delta}$ , we get

$$\begin{aligned} \partial_a \eta_{123} = \eta_{\alpha\beta\gamma\delta} & \left[ \varphi_a \ell^\alpha e_1^\beta e_2^\gamma e_3^\delta + \Gamma_{a1}^1 \ell^\alpha e_1^\beta e_2^\gamma e_3^\delta + \Gamma_{a2}^2 \ell^\alpha e_1^\beta e_2^\gamma e_3^\delta + \right. \\ & \left. \Gamma_{a3}^3 \ell^\alpha e_1^\beta e_2^\gamma e_3^\delta \right] = \eta_{123} (\varphi_a + \Gamma_{ac}^c) = \eta_{123} \Gamma_a \end{aligned} \quad (31)$$

so that being  $\eta_{123}$  positive everywhere we find  $\eta_{123} = C e^\phi$  where  $C$  is a positive constant.

Let us change now the normalization factor of the normal form, without changing the direction of the rigging vector  $\vec{\ell}$  anywhere. So, we put  $n'_\mu = \sigma n_\mu$  and if we want to keep the volume form positive, the factor  $\sigma$  must be positive everywhere. The changes induced by this transformation are

$$\begin{aligned} \eta'_{123} &= \frac{1}{\sigma} \eta_{123}, \quad K'_{ab} = \sigma K_{ab}, \quad \vec{\ell}' = \frac{1}{\sigma} \vec{\ell}, \quad \Psi'^a_b = \frac{1}{\sigma} \Psi^a_b \\ \omega^a_\mu &\text{ and } \Gamma^a_{bc} \text{ remain invariant, } \varphi'_a = \varphi_a - \partial_a \log \sigma, \quad \Gamma'_a = \Gamma_a - \partial_a \log \sigma. \end{aligned} \quad (32)$$

It is a well-known fact in differential geometry that in an affine manifold (that is, with a linear connection) possessing a well-defined volume form  $\boldsymbol{\eta}$ , the Stokes theorem can be rewritten as a Gauss theorem

$$\int_U \nabla_\mu X^\mu \boldsymbol{\eta} = \int_{\partial U} X^\mu d\sigma_\mu,$$

if and only if the connection is volume preserving, that is to say  $\nabla \boldsymbol{\eta} = 0$ . In the previous formula  $U$  is an open neighbourhood in the manifold,  $X^\mu$  is any vector field defined at least on  $U$  and  $d\sigma_\mu$  is the normal volume form defined on the boundary  $\partial U$  of  $U$ . Of course, for all metric connections this is immediately true and the Gauss theorem holds in the spacetime. In our case, however, we have defined in the hypersurface a volume form and a connection that are not metric and then we are not sure that the Gauss theorem holds in the hypersurface. Therefore, we want to ascertain under which conditions the rigged connections in the hypersurface are volume preserving. To that aim, let us calculate

$$\overline{\nabla}_a \eta_{bcd} = \partial_a \eta_{bcd} - \Gamma_{ba}^e \eta_{ecd} - \Gamma_{ca}^e \eta_{bed} - \Gamma_{da}^e \eta_{bce} = \partial_a \eta_{bcd} - \Gamma_{ea}^e \eta_{bcd}$$

where we have used the antisymmetry property of  $\boldsymbol{\eta}$ . Making use here of formula (31) we can write then

$$\overline{\nabla}_a \eta_{bcd} = \varphi_a \eta_{bcd} . \quad (33)$$

In the same way, it is easy to find the analogous expression for the contravariant volume form:

$$\bar{\nabla}_a \eta^{bcd} = -\varphi_a \eta^{bcd} . \quad (34)$$

Consequently, we have proven the following standard result.

**Theorem 2** *Given a normal form and a rigging vector, the necessary and sufficient condition for the rigged connection to be volume preserving is that  $\varphi = 0$ .*

The condition  $\varphi = 0$  is called by Schouten [6] the *second condition of normalization*.

Looking now at the transformation formula (32) for  $\varphi$ , we see that the above theorem can be improved if we allow for changes in the normalization of the normal form (or the volume element). In this sense, it is immediate the following theorem.

**Theorem 3** *Given a rigging direction, the necessary and sufficient condition such that there exists a normalization of the normal form (or equivalently a volume form in the hypersurface) in which the rigged connection is volume preserving is that the 1-form  $\varphi$  be exact. Moreover, this volume form, if it exists, is unique except for a constant factor.*

The question of whether or not there always exists a rigging and a normalization factor of the normal form such that the rigged connection is volume preserving is a little bit more complicated. In the case where the second fundamental form is not degenerate (or even degenerate only once), the existence of such a connection can be shown without difficulty. In other cases, however, that depends on some integrability conditions involving both the manifold and hypersurface structures. In the next section we present another type of connection in the hypersurface which is always volume preserving.

## 4 Second Connection in a General Hypersurface: The Rigged Metric Connection.

We are now going to develop a completely different method to define a connection in the hypersurface. As before, we begin with a rigging vector field  $\vec{\ell}$  defined on every point in  $\Sigma$  which verifies  $\mathbf{n}(\vec{\ell}) = 1$ . We have already studied  $\Phi^*(g)$  and we have shown that the first fundamental form is degenerate if and only if the normal form is null. We consider now the symmetric non-degenerate tensor  $g^{\mu\nu}$  and its projection into the hypersurface by means of the map  $T$ . Its components are

$$g^{ab} = g^{\mu\nu} \omega_\mu^a \omega_\nu^b$$



and obviously they do depend on the rigging of the hypersurface. With this new tensor in the hypersurface we can complete relations (4) with the following formulas

$$g^{ab}\ell_b = -(\vec{\ell} \cdot \vec{\ell})n^a, \quad g^{ab}\bar{g}_{bc} = \delta_c^a - n^a\ell_c. \quad (35)$$

Let us find a necessary and sufficient condition for this symmetric contravariant tensor to be non-degenerate. First of all we establish the following lemma.

**Lemma 3** *The rigging vector  $\vec{\ell}$  can be expressed in the following form*

$$\ell^\mu = -\eta_{123}\eta^{\mu\nu\rho\lambda}\omega_\nu^1\omega_\rho^2\omega_\lambda^3$$

*Proof:* The proportionality between  $\ell^\mu$  and the vector  $\chi^\mu \equiv \eta^{\mu\nu\rho\lambda}\omega_\nu^1\omega_\rho^2\omega_\lambda^3$  follows from the relation  $\omega_\mu^a\chi^\mu = 0$  for every  $a$ , so we can write  $\ell^\mu = B\eta^{\mu\nu\rho\lambda}\omega_\nu^1\omega_\rho^2\omega_\lambda^3$  for some factor  $B$ . Using now that  $\ell^\mu n_\mu = 1$  and formula (1) we have

$$1 = n_\mu\ell^\mu = \frac{B}{\eta_{123}}\eta_{\mu\beta\gamma\delta}\eta^{\mu\nu\rho\lambda}e_1^\beta e_2^\gamma e_3^\delta \omega_\nu^1\omega_\rho^2\omega_\lambda^3 = -\frac{B}{\eta_{123}}$$

and therefore  $B = -\eta_{123}$ , as we wanted to prove.

An easy calculation using the above expression for  $\vec{\ell}$  shows the following result

$$\ell^\alpha\ell_\alpha = -(\eta_{123})^2 \det(g^{ab}) \quad (36)$$

which is similar to formula (2), and then, analogously to Lemma 1, we have:

**Proposition 4**  *$g^{ab}$  is degenerate in a point  $x \in \Sigma$  if and only if the rigging vector is null at that point.*

There is, though, an important difference between the usefulness of this result and that of the similar one for the normal form  $\mathbf{n}$ . Whereas the normal form  $\mathbf{n}$  is determined by the hypersurface, the rigging vector  $\vec{\ell}$  can be chosen in many different ways and, in particular, it can be taken non-null everywhere in  $\Sigma$  so that the tensor  $g^{ab}$  is non-degenerate everywhere in the hypersurface. In this case, we have that the tensor  $g^{ab}$  has an inverse, that we will call  $\gamma_{ab}$ , satisfying  $g^{ab}\gamma_{bc} = \delta_c^a$ . It is easy to see that this tensor can be written

$$\gamma_{ab} = \bar{g}_{ab} - \frac{1}{(\vec{\ell} \cdot \vec{\ell})}\ell_a\ell_b. \quad (37)$$

Of course, there are many symmetric two-covariant tensors in the manifold such that  $\Phi^*$  applied to them gives  $\gamma_{ab}$ , but we have already defined a privileged one among them,

which we will call  $\gamma_{\mu\nu}$ , by demanding that  $\ell^\mu\gamma_{\mu\nu} = 0$  or, in other words, by applying  $\Lambda$  to  $\gamma_{ab}$ . This tensor is

$$\gamma_{\mu\nu} = g_{\mu\nu} - \frac{1}{(\vec{\ell} \cdot \vec{\ell})} \ell_\mu \ell_\nu \quad (38)$$

from where we can guess a possible physical interpretation of this construction. The form of the tensor (38) suggests that we can take the rigging vector as a timelike vector in the manifold, or equivalently, as an *observer*, and then we define for every hypersurface in the manifold a non-degenerate two-covariant symmetric tensor by restricting the metric in the canonical three-spaces of the observer to the hypersurface. The freedom in choosing the rigging is simply then the possibility of changing the observer in spacetime. Thus, different riggings induce different tensors of type (37) in a given hypersurface, but this would just represent the fact that different observers would “see” the same hypersurface with different metric properties. This possibility was in fact suggested some years ago, in a somewhat different form and for a very particular case, in [13].

From a mathematical point of view, however, the important thing is that we have defined a symmetric two-covariant non-degenerate tensor in the hypersurface, and thereby we have the unique metric connection associated to it, given by

$$\tilde{\Gamma}_{bc}^a = \frac{1}{2} g^{ad} (\partial_b \gamma_{dc} + \partial_c \gamma_{db} - \partial_d \gamma_{bc}) \quad (39)$$

that we call the *rigged metric connection*.

Let us now try to relate the rigged connection and the rigged metric connection defined by the same (non-null) rigging. Because of the relation  $\bar{\nabla}_a \gamma_{bc} = \partial_a \gamma_{bc} - \Gamma_{ba}^e \gamma_{ec} - \Gamma_{ca}^e \gamma_{be}$ , we can rewrite the formula defining  $\tilde{\Gamma}_{bc}^a$  as

$$\tilde{\Gamma}_{bc}^a = \frac{1}{2} g^{ad} (\bar{\nabla}_b \gamma_{dc} + \bar{\nabla}_c \gamma_{db} - \bar{\nabla}_d \gamma_{bc}) + \Gamma_{bc}^a \quad (40)$$

which makes explicit the following obvious statement:

**Proposition 5** *Given a rigging, the two connections  $\tilde{\Gamma}$  and  $\Gamma$  coincide iff  $\bar{\nabla}_d \gamma_{bc} = 0$ .*

We are going to reexpress this result in a more interesting and useful form by using expression (37) and expanding  $\bar{\nabla}_a \gamma_{bc}$ . We only need to know the covariant derivatives of  $\bar{g}_{ab}$  and  $l_c$  (formulas (25) and (26)), as well as the differential of the norm of the rigging,  $\partial_c (\vec{\ell} \cdot \vec{\ell})$ . This can be found straightforwardly from formula (21)

$$\partial_c (\vec{\ell} \cdot \vec{\ell}) = 2 (\vec{\ell} \cdot \vec{\ell}) \varphi_c + 2 \Psi_c^e \ell_e \quad (41)$$

and putting all this together we easily find

$$\bar{\nabla}_d \gamma_{bc} = -\frac{1}{(\vec{\ell} \cdot \vec{\ell})} \Psi_d^e (\gamma_{ec} \ell_b + \gamma_{eb} \ell_c) .$$

**Theorem 4** *Given a non-null rigging, the necessary and sufficient condition such that the rigged connection  $\Gamma$  and the rigged metric connection  $\tilde{\Gamma}$  coincide is that, for any point  $p \in \Sigma$ , either  $\Psi_b^a|_p = 0$  or  $\ell_a|_p = 0$ .*

*Proof:* The necessary and sufficient condition is that  $\bar{\nabla}_a \gamma_{bc} = 0$  or, what is the same,  $\Psi_c^e (\gamma_{ea} \ell_b + \gamma_{eb} \ell_a) = 0$ . But we know that if an object of type  $U_a V_b + U_b V_a$  vanishes then necessarily  $U_a = 0$  or  $V_a = 0$ . Applying this result to the previous expression, and noting that the index  $c$  plays no role in this reasoning, we have  $\Psi_c^e \gamma_{ea} = 0$  or  $\ell_a = 0$ , and as the metric  $\gamma_{ab}$  is not degenerate the theorem follows.

From its definition, it is trivial that  $\ell_c = 0$  if and only if  $\vec{\ell}$  is proportional to the vector  $\vec{n}$  ( $\ell_a = 0 \Leftrightarrow \ell_\mu e_a^\mu = 0 \forall a \Leftrightarrow \ell_\alpha$  is proportional to  $n_\alpha$ ). In the case of hypersurfaces non-null everywhere we can choose the normal vector  $\vec{n}$  as the rigging vector and then we have  $\ell_a = n_a = 0$ . With this choice of the rigging the two connections coincide, as is obvious from its construction, giving the natural connection in the hypersurface. In the general case, however, if there is some point where the hypersurface is null, we cannot take the normal vector as the rigging and then  $\ell_a \neq 0$  for some  $a$ . So, in the general case, the only possibility for having a rigging such that the two connections coincide is that  $\Psi_b^a = 0$ . In consequence, we are now going to study under which conditions one can choose a rigging such that  $\Psi_c^a = 0$  everywhere in the hypersurface. Recalling expression (21) we have that  $\Psi_b^a = 0$  is equivalent to

$$e_b^\nu \nabla_\nu \ell^\mu = \varphi_b \ell^\mu \quad (42)$$

and then contracting with vector  $\vec{\ell}$  we find

$$\partial_b \left( \log \left( \sqrt{|\vec{\ell} \cdot \vec{\ell}|} \right) \right) = \varphi_b$$

so that a necessary condition for the equation (42) to hold is that  $\varphi$  be an exact 1-form. We can now define a new vector field

$$\vec{V} \equiv \frac{\vec{\ell}}{\sqrt{|\vec{\ell} \cdot \vec{\ell}|}}$$

and it is very easy to check that if the vector  $\vec{\ell}$  verifies equation (42), then the vector  $\vec{V}$  verifies

$$e_b^\nu \nabla_\nu V^\mu = 0 \quad (43)$$

and conversely, if  $\vec{V}$  verifies equation (43) then any vector of the form  $\vec{\ell} \equiv \sigma \vec{V}$  verifies equation (42), where  $\sigma$  is any non-vanishing function in the hypersurface. This last

equation is written in normal form, and then we can study its integrability conditions by the standard procedure. These integrability conditions are found to be

$$R_{\sigma\alpha\beta}^{\mu} e_c^{\alpha} e_b^{\beta} V^{\sigma} = 0 \quad (44)$$

When this relation is identically satisfied, that is to say when  $R_{\sigma\alpha\beta}^{\mu} e_c^{\alpha} e_b^{\beta} = 0$ , then equation (43) has a general solution depending on three constant parameters (in principle there are four constants that are the initial conditions  $V^{\mu}|_p$  at some point  $p \in \Sigma$ , but they are subject to the relation  $V^{\mu} \cdot V_{\mu}|_p = \pm 1$ ). If they are not identically verified, we must go on with the integrability conditions, which are successively

$$e_d^{\rho} \nabla_{\rho} \left( R_{\sigma\alpha\beta}^{\mu} e_c^{\alpha} e_b^{\beta} \right) V^{\sigma} = 0 \quad (45)$$

$$e_f^{\kappa} \nabla_{\kappa} \left( e_d^{\rho} \nabla_{\rho} \left( R_{\sigma\alpha\beta}^{\mu} e_c^{\alpha} e_b^{\beta} \right) \right) V^{\sigma} = 0 \quad (46)$$

$$e_g^{\lambda} \nabla_{\lambda} \left( e_f^{\kappa} \nabla_{\kappa} \left( e_d^{\rho} \nabla_{\rho} \left( R_{\sigma\alpha\beta}^{\mu} e_c^{\alpha} e_b^{\beta} \right) \right) \right) V^{\sigma} = 0 \quad (47)$$

⋮

If the first equation in the row is verified identically (and the previous integrability condition is not) then the general solution has as many arbitrary constants of integration as the number of solutions (if any) for  $V^{\alpha}$  of the equation (44) at any point in the hypersurface. The following integration conditions must be understood in a similar form. It is evident that the existence or not of a rigging whose two connections coincide depends both on the form of the hypersurface and on the manifold in which it is imbedded.

To end this section, we shall establish the relation between  $\tilde{\Gamma}_{ac}^a$  and  $\Gamma_{ac}^a$ . Recalling formula (36):  $(\vec{\ell} \cdot \vec{\ell}) = -(\eta_{123})^2 \det(g^{ab}) = -\frac{(\eta_{123})^2}{\det(\gamma_{ab})}$  and using the fact that  $\tilde{\Gamma}_{ac}^a = \partial_c \left( \log \left( \sqrt{|\det(\gamma_{ab})|} \right) \right)$  we find

$$\tilde{\Gamma}_{ac}^a = \Gamma_{ac}^a + \varphi_c - \frac{1}{2} \partial_c \left( \log \left( |\vec{\ell} \cdot \vec{\ell}| \right) \right) = \Gamma_c - \frac{1}{2} \partial_c \left( \log \left( |\vec{\ell} \cdot \vec{\ell}| \right) \right) \quad (48)$$

which is the desired relation. These formulas lead us to the following proposition.

**Proposition 6** *The volume preserving connection  $\tilde{\Gamma}$  is such that  $\tilde{\Gamma}_{ac}^a = \Gamma_c$  when  $(\vec{\ell} \cdot \vec{\ell})$  is chosen to be constant in the hypersurface.*

Therefore, the rigged metric connections have some advantages like being always volume preserving and the fact that each one fixes a unique volume element in the hypersurface. They are also good connections from the physical point of view because, as explained above, they can be interpreted as connections associated with observers in space-time. Nevertheless, the geometrical meaning of the rigged metric connections is obscure, contrarily to what happens with the rigged connections which are defined

simply by decomposing the tangent spaces and then projecting to the hypersurface (with respect to the rigging). In any case, both constructions provide a geometrical structure to general hypersurfaces *imbedded in spacetime*, and they will also allow us to define the proper junction conditions in general relativity for arbitrary hypersurfaces of discontinuity, regardless of its time-, space- or light-like character, which can also vary from point to point. This is the subject of the next section.

## 5 Junction Conditions.

Consider now two  $C^3$  spacetimes  $V^+$  and  $V^-$ , each of them with boundary  $\Sigma^+$  and  $\Sigma^-$  and  $C^2$  metrics  $g^+$  and  $g^-$ . Assume further that there is a  $C^3$  diffeomorphism from  $\Sigma^-$  to  $\Sigma^+$ . This is the typical situation in General Relativity where the glueing of two spacetimes by means of identification of points on the boundaries is considered in order to study their possible posterior matching. However, as pointed out by Clarke and Dray [21], the mere identification of points in  $\Sigma^+$  and  $\Sigma^-$  does *not* by itself give a well defined geometry in the sense that one should also specify how the tangent spaces are to be identified. To clarify this, let us define  $V_4$ , the whole spacetime, as the disjoint union of  $V^+$  and  $V^-$  with diffeomorphically related points in  $\Sigma^+$  and  $\Sigma^-$  identified. On the complementary of  $\Sigma^\pm$  in  $V_4$  we have the metric  $g$  given by  $g^+$  in  $V^+$  and  $g^-$  in  $V^-$ . The main result proven by Clarke and Dray reads as follows.

**Theorem 5** *Under the above assumptions, there exists a unique  $C^1$  atlas on  $V_4$  which induces the given  $C^3$  structures on  $V^+$  and  $V^-$  and such that  $g$  admits a continuous extension to the whole  $V_4$  (and which is maximal with respect to these properties) if and only if  $\Sigma^+$  and  $\Sigma^-$  are isometrical with respect to their first fundamental forms inherited from  $V^+$  and  $V^-$ ; that is to say, if and only if their respective first fundamental forms  $\bar{g}^+$  and  $\bar{g}^-$  agree.*

To be precise, Clarke and Dray proved this theorem under the added assumption that the signatures of  $\bar{g}^+$  and  $\bar{g}^-$  are constant. This assumption is superfluous, though, and their proof can be easily generalized to arbitrary hypersurfaces in which the signature can change from point to point.

The above theorem is of great importance, because if one wishes to define Einstein's equations, even in the distributional sense, it is necessary that the metric of the spacetime be, at least, continuous. Thus, if we consider the whole spacetime  $V_4$  and denote simply by  $\Sigma$  the image of  $\Sigma^+$  or  $\Sigma^-$  in it, the necessary and sufficient condition such that the glued spacetime  $V_4$  has well-defined Einstein's equations for the metric  $g$  is that, in a given coordinate system of  $\Sigma$ , the first fundamental form of  $\Sigma$  calculated from  $V^+$  coincide with the first fundamental form of  $\Sigma$  calculated from  $V^-$ :

$$\bar{g}^+ = \bar{g}^- . \tag{49}$$

We call relations (49) the *preliminary junction conditions*. In a practical problem, these conditions work as follows. We are given two imbeddings  $x_{\pm}^{\mu} = x_{\pm}^{\mu}(\xi^a)$  of  $\Sigma$ , where  $x_{\pm}^{\mu}$  are local coordinates for  $V^{\pm}$ , respectively, and  $\xi^a$  are intrinsic coordinates for  $\Sigma$ . Therefore, we also have the vectors tangent to  $\Sigma$ :  $\vec{e}_a^{\pm}$ . But if the preliminary junction conditions hold, then we have in the coordinate system  $\{\xi^a\}$

$$\bar{g}_{ab}^+ = \bar{g}_{ab}^-$$

or, equivalently, the scalar products of the vectors  $\vec{e}_a^{\pm}$  coincide from  $V^+$  and  $V^-$ . There only remains to choose the riggings  $\vec{\ell}^{\pm}$  such that  $\{\vec{\ell}^{\pm}, \vec{e}_a^{\pm}\}$  are both bases with the same orientation satisfying

$$\ell_a^+ = \ell_a^- \quad (\ell \cdot \ell)^+ = (\ell \cdot \ell)^-$$

and then identify the bases in the tangent spaces  $\{\vec{\ell}^+, \vec{e}_a^+\} \equiv \{\vec{\ell}^-, \vec{e}_a^-\} \equiv \{\vec{\ell}, \vec{e}_a\}$  by *definition* dropping the  $\pm$ . In the resulting spacetime, and due to the above theorem, there exists a unique structure with coordinate systems such that the metric  $g$  of  $V_4$  is continuous, hence, the components  $\ell^{\mu}$  and  $e_a^{\mu}$  in these coordinate systems are well defined.

From now on, we assume that the preliminary junction conditions hold such that the above construction has been carried out and the whole spacetime  $(V_4, g)$  has a hypersurface  $\Sigma$  splitting the manifold into the two open sets  $V^-$  and  $V^+$ , whose boundary is  $\Sigma$ , and such that the metric tensor  $g$  is continuous on the whole manifold and at least of type  $C^2$  in both  $V^-$  and  $V^+$ . We shall also assume that the derivatives up to second order of this tensor field have a well-defined limit on the hypersurface of separation  $\Sigma$  coming from both  $V^-$  and  $V^+$ . We will not restrict the type of the hypersurface in any way whatsoever and, consequently, we will use the theory of general hypersurfaces developed above. Our aim is to obtain the curvature tensors of such a spacetime and thereby to find the necessary and sufficient junction conditions which forbid the existence of singular parts in the curvature (sometimes these singular parts are called surface layers or impulsive gravitational waves). The natural way to study this sort of problems is by using the theory of tensor distributions on manifolds. For a brief summary and for notations used from here on, see the appendix in this paper.

First of all, we write the step on  $\Sigma$  of the derivative of a function  $f$ . It is not difficult to see that, for every vector  $\vec{V}$  tangent to the hypersurface, we have

$$V^{\mu} [\partial_{\mu} f] = V^{\mu} \partial_{\mu} [f]$$

and therefore, using the basis  $\{\mathbf{n}, \omega^a\}$  of the dual tangent plane, we obtain

$$[\partial_{\mu} f] = A n_{\mu} + \omega_{\mu}^a \partial_a [f] \quad (50)$$

where  $A$  is a scalar function on  $\Sigma$  defined by  $A \equiv \ell^{\mu} [\partial_{\mu} f]$ . Obviously,  $A$  depends on the rigging, but formula (50) does not. Thus, when the function  $f$  is continuous across

$\Sigma$ ,  $A$  is independent of the rigging. This is what happens with the metric tensor itself, and from the previous equation it follows that

$$[\partial_\beta g_{\lambda\rho}] = \zeta_{\lambda\rho} n_\beta \quad (51)$$

where  $\zeta_{\lambda\rho}$  is a two-covariant symmetric tensor field defined on the hypersurface  $\Sigma$  and independent of the rigging. Using this formula it is immediate to find

$$[\Gamma_{\beta\gamma}^\alpha] = \frac{1}{2} \left( \zeta_\lambda^\alpha n_\beta + \zeta_\beta^\alpha n_\lambda - n^\alpha \zeta_{\beta\lambda} \right) , \quad (52)$$

and substituting this in formula (115) of the appendix, we get

$$\underline{R}_{\beta\lambda\mu}^\alpha = (\underline{1} - \underline{\theta}) \cdot R_{\beta\lambda\mu}^{-\alpha} + \underline{\theta} \cdot R_{\beta\lambda\mu}^{+\alpha} + \delta \cdot H_{\beta\lambda\mu}^\alpha \quad (53)$$

where  $H_{\beta\lambda\mu}^\alpha$  is a tensor called *the singular part of the Riemann tensor distribution* and is defined only on the hypersurface as follows

$$H_{\beta\lambda\mu}^\alpha \equiv n_\lambda [\Gamma_{\beta\mu}^\alpha] - n_\mu [\Gamma_{\beta\lambda}^\alpha] = \frac{1}{2} \left\{ n^\alpha (\zeta_{\beta\lambda} n_\mu - \zeta_{\beta\mu} n_\lambda) + n_\beta (\zeta_\mu^\alpha n_\lambda - \zeta_\lambda^\alpha n_\mu) \right\} . \quad (54)$$

Of course, this tensor has the algebraic properties of a Riemann tensor. For the Ricci tensor distribution, the analogous relation is

$$\underline{R}_{\beta\mu} = (\underline{1} - \underline{\theta}) \cdot R_{\beta\mu}^- + \underline{\theta} \cdot R_{\beta\mu}^+ + \delta \cdot H_{\beta\mu} \quad (55)$$

where we have introduced its singular part

$$H_{\beta\mu} \equiv H_{\beta\alpha\mu}^\alpha = \frac{1}{2} (n^\alpha \zeta_{\alpha\beta} n_\mu + n^\alpha \zeta_{\alpha\mu} n_\beta - n^\alpha n_\alpha \zeta_{\beta\mu} - \zeta_\alpha^\alpha n_\beta n_\mu) \quad (56)$$

which is a symmetric tensor defined only at points of  $\Sigma$ . With regard to the scalar curvature distribution, we also have

$$\underline{R} = (\underline{1} - \underline{\theta}) \cdot R^+ + \underline{\theta} \cdot R^- + \delta \cdot H \quad (57)$$

where its singular part, defined only on the hypersurface, is given by

$$H \equiv g^{\beta\mu} \Big|_\Sigma H_{\beta\mu} = n^\alpha n^\beta \zeta_{\alpha\beta} - n^\alpha n_\alpha \zeta_\beta^\beta . \quad (58)$$

Finally, it is interesting to find a similar expression for the Einstein tensor distribution, because of its close relation with the matter contents of the spacetime. Using the previous formulas, the definition  $G_{\beta\mu} = R_{\beta\mu} - \frac{1}{2} g_{\beta\mu} R$  allows us to write

$$\underline{G}_{\beta\mu} = (\underline{1} - \underline{\theta}) \cdot G_{\beta\mu}^- + \underline{\theta} \cdot G_{\beta\mu}^+ + \delta \cdot \tau_{\beta\mu} \quad (59)$$

where we have defined a new symmetric tensor on  $\Sigma$  as

$$\begin{aligned} \tau_{\beta\mu} \equiv H_{\beta\mu} - \frac{1}{2} g_{\beta\mu}|_{\Sigma} H = \frac{1}{2} \{ & n^\alpha \zeta_{\alpha\beta} n_\mu + n^\alpha \zeta_{\alpha\mu} n_\beta - n^\alpha n_\alpha \zeta_{\beta\mu} - \zeta_\alpha^\alpha n_\beta n_\mu \\ & - g_{\beta\mu}|_{\Sigma} (n^\alpha n^\rho \zeta_{\alpha\rho} - n^\alpha n_\alpha \zeta_\rho^\rho) \}. \end{aligned} \quad (60)$$

Contracting the last expression with the normal vector to the hypersurface  $n^\mu$  we find that

$$\tau_{\beta\mu} n^\mu = 0 \quad . \quad (61)$$

Using the projection tensor  $P_\beta^\alpha$ , we can decompose the tensor  $\zeta_{\beta\mu}$ , defined at every point of the hypersurface, into its tangent part (with respect to the rigging vector  $\vec{\ell}$ ) and its rigged part as

$$\zeta_{\beta\mu} = \zeta_{\beta\mu}^{\vec{\ell}} + \zeta_\beta^{\vec{\ell}} n_\mu + \zeta_\mu^{\vec{\ell}} n_\beta + \zeta^{\vec{\ell}} n_\beta n_\mu \quad (62)$$

where we have defined

$$\zeta_{\beta\mu}^{\vec{\ell}} \equiv P_\beta^\lambda P_\mu^\alpha \zeta_{\lambda\alpha}, \quad \zeta_\beta^{\vec{\ell}} \equiv \ell^\lambda P_\beta^\rho \zeta_{\lambda\rho}, \quad \zeta^{\vec{\ell}} \equiv \ell^\lambda \ell^\mu \zeta_{\lambda\mu} \quad . \quad (63)$$

Here, the first two objects are tangent to the rigged hypersurface in the sense that they are orthogonal to the rigging vector. Therefore, they are isomorphically related with tensors defined in the hypersurface through the maps  $\Phi^*$  and  $\Lambda$  of section 2. The important point now is that substituting the decomposition (62) of  $\zeta$  in expression (54) we find

$$H_{\alpha\beta\lambda\mu} = \frac{1}{2} \{ n_\alpha (\zeta_{\beta\lambda}^{\vec{\ell}} n_\mu - \zeta_{\beta\mu}^{\vec{\ell}} n_\lambda) + n_\beta (\zeta_{\alpha\mu}^{\vec{\ell}} n_\lambda - \zeta_{\alpha\lambda}^{\vec{\ell}} n_\mu) \} \quad (64)$$

so that only the tangent part  $\zeta_{\beta\mu}^{\vec{\ell}}$  of  $\zeta_{\beta\mu}$  appears in the singular part of the Riemann tensor distribution.

Next, we shall find an intrinsic expression for  $\zeta_{\beta\mu}^{\vec{\ell}}$  depending only on the tensor  $\mathcal{H}_{ab}$ . To that end, let us recall that its definition is  $\mathcal{H}_{ab} = e_a^\nu e_b^\mu \nabla_\nu \ell_\mu$  and, consequently, being the connection discontinuous across  $\Sigma$ ,  $\mathcal{H}_{ab}$  will be different when coming from  $V^+$  or from  $V^-$ . Although this object is defined only in the hypersurface and then it cannot be continuous nor discontinuous, we will denote by  $[\mathcal{H}_{ab}]$  the difference at each point in  $\Sigma$  of  $\mathcal{H}_{ab}$  defined with the connection of  $V^+$  and  $\mathcal{H}_{ab}$  defined with the connection of  $V^-$ .<sup>1</sup> Making use of (52) this difference tensor becomes

$$[\mathcal{H}_{ab}] = -\ell_\mu [\Gamma_{\sigma\nu}^\mu] e_a^\nu e_b^\sigma = -\frac{1}{2} \ell_\mu (\zeta_\sigma^\mu n_\nu + \zeta_\nu^\mu n_\sigma - n^\mu \zeta_{\sigma\nu}) e_a^\nu e_b^\sigma = \frac{1}{2} \zeta_{\sigma\nu} e_a^\nu e_b^\sigma \quad (65)$$

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<sup>1</sup>For other intrinsic objects of the hypersurface we will use also the brackets to denote the difference between the objects defined form  $V^+$  and  $V^-$ . Abusing the language, we will name ‘continuous’ such objects with vanishing difference.



where we have taken into account that the rigging form  $\ell_\mu$ , the tangent vectors  $e_a^\mu$  and  $e_a^\nu \partial_\nu e_b^\mu$  have the same value from  $V^-$  or  $V^+$ . Note that  $[\mathcal{H}_{ab}]$  is symmetric despite the fact that  $\mathcal{H}_{ab}$  itself is not.  $[\mathcal{H}_{ab}]$  is uniquely related, through the map  $\Lambda$ , with a symmetric two-covariant tensor field in the manifold defined only at points on the hypersurface. This tensor, which we will denote as before with the same symbol  $[\mathcal{H}_{\mu\nu}]$ , is

$$[\mathcal{H}_{\mu\nu}] = \frac{1}{2} \zeta_{\sigma\rho} P_\mu^\sigma P_\nu^\rho = \frac{1}{2} \zeta_{\mu\nu}^{\vec{\ell}}$$

and, of course, it satisfies

$$[\mathcal{H}_{\mu\nu}] \ell^\mu = 0. \quad (66)$$

Therefore, we obtain for the singular part of the Riemann tensor distribution

$$H_{\beta\lambda\mu}^\alpha = n^\alpha (-[\mathcal{H}_{\beta\mu}] n_\lambda + [\mathcal{H}_{\beta\lambda}] n_\mu) + n_\beta (-[\mathcal{H}_\lambda^\alpha] n_\mu + [\mathcal{H}_\mu^\alpha] n_\lambda) \quad (67)$$

and this expression allows us to prove the following fundamental theorem

**Theorem 6** *The singular part of the Riemann tensor distribution vanishes if and only if  $[\mathcal{H}_{\mu\nu}] = 0$ , or equivalently, iff  $[\mathcal{H}_{ab}] = 0$ .*

*Proof:*

[ $\Rightarrow$ ] If  $[\mathcal{H}_{\alpha\beta}] = 0$  then from the previous formula  $H_{\beta\lambda\mu}^\alpha = 0$ .

[ $\Leftarrow$ ] Suppose now that  $H_{\beta\lambda\mu}^\alpha = 0$ . Contracting then (67) with  $\ell^\beta \ell^\lambda$  and making use of (66) we obtain  $0 = \ell^\beta \ell^\lambda H_{\beta\lambda\mu}^\alpha = [\mathcal{H}_\mu^\alpha]$  and the theorem follows.

From now on, we shall refer to  $[\mathcal{H}_{\mu\nu}] = 0$  as *the junction conditions*. These conditions assure that all the curvature (or matter) tensors have, at most, finite discontinuities across  $\Sigma$ . Furthermore, when the junction conditions are satisfied we get from (51) and (62) a structure for the discontinuities of the first derivatives of the metric tensor which allows us to perform a  $C^1$  change of coordinates such that the metric becomes  $C^1$ , in accordance with the minimal differentiability requirements of Lichnerowicz [15]. In order to see this, let us note that under  $C^1$  change of coordinates,  $x^{\alpha'} = x^{\alpha'}(x^\beta)$  the discontinuity of the partial derivative of the jacobian matrix of the transformation reads

$$\left[ \frac{\partial}{\partial x^{\mu'}} \left( \frac{\partial x^\alpha}{\partial x^{\alpha'}} \right) \right] = n_{\mu'} n_{\alpha'} B^\alpha \quad (68)$$

where  $B^\alpha$  is a vector defined on the hypersurface. Hence it is easy to check from the transformation law for the discontinuities of the first derivatives of the metric that if we choose a coordinate change satisfying  $B_\alpha = -\left(\zeta_\alpha^{\vec{\ell}} + \frac{1}{2}\zeta^{\vec{\ell}} n_\alpha\right)$  then the metric becomes  $C^1$  in the new coordinates.

Similar results can be proven for the singular parts of the Ricci tensor, the scalar curvature and the Einstein tensor distributions. The expressions for these singular parts come directly from (67) and read, respectively

$$H_{\beta\mu} = -(\vec{n} \cdot \vec{n}) [\mathcal{H}_{\beta\mu}] + n^\alpha [\mathcal{H}_{\alpha\beta}] n_\mu + n^\alpha [\mathcal{H}_{\alpha\mu}] n_\beta - [\mathcal{H}_\alpha^\alpha] n_\beta n_\mu \quad , \quad (69)$$

$$H = -2(\vec{n} \cdot \vec{n}) [\mathcal{H}_\alpha^\alpha] + 2n^\alpha n^\beta [\mathcal{H}_{\alpha\beta}] \quad , \quad (70)$$

$$\begin{aligned} \tau_{\beta\mu} = & -(\vec{n} \cdot \vec{n}) [\mathcal{H}_{\beta\mu}] + n^\alpha [\mathcal{H}_{\alpha\beta}] n_\mu + n^\alpha [\mathcal{H}_{\alpha\mu}] n_\beta - [\mathcal{H}_\alpha^\alpha] n_\beta n_\mu \\ & - g_{\beta\mu}|_\Sigma (-(\vec{n} \cdot \vec{n}) [\mathcal{H}_\alpha^\alpha] + n^\alpha n^\nu [\mathcal{H}_{\alpha\nu}]) \quad . \end{aligned} \quad (71)$$

Summarizing, we have the following general theorem.

**Theorem 7** *1. At a point  $x \in \Sigma$  where the hypersurface is not null, the singular part of the Ricci tensor distribution vanishes if and only if  $[\mathcal{H}_{\beta\mu}] = 0$ , hence, iff the singular part of the Riemann tensor distribution vanishes.*

*2. At a point  $x \in \Sigma$  where the hypersurface is null, the singular part of the Ricci tensor distribution vanishes if and only if  $n^\alpha [\mathcal{H}_{\alpha\beta}] = 0$  and  $[\mathcal{H}_\alpha^\alpha] = 0$ .*

*3. The singular part of the energy-momentum tensor distribution vanishes if and only if so does the singular part of the Ricci tensor distribution.*

*4. The singular part of the curvature scalar distribution vanishes if and only if  $(\vec{n} \cdot \vec{n}) [\mathcal{H}_\alpha^\alpha] = [\mathcal{H}_{\beta\mu}] n^\beta n^\mu$ .*

*Proof:* For the first two assertions, if  $H_{\beta\mu} = 0$  then we can contract equation (69) with  $\ell^\beta$  and  $\ell^\mu$  to obtain  $[\mathcal{H}_\alpha^\alpha] = 0$ , and contracting now the same equation only with  $\ell^\beta$  and using this last result we have  $n^\alpha [\mathcal{H}_{\alpha\mu}] = 0$ . Thus,  $H_{\beta\mu} = -(\vec{n} \cdot \vec{n}) [\mathcal{H}_{\beta\mu}] = 0$  so that the direct part of the theorem follows. The converse is trivial from equation (69) itself. The third assertion is then immediate from the definition  $\tau_{\beta\mu} = H_{\beta\mu} - \frac{1}{2} g_{\beta\mu}|_\Sigma H$ . Finally, the last part is a simple consequence of equation (70).

We see from this theorem that, at points where  $\Sigma$  is not null, the vanishing of the matter singular part is equivalent to the vanishing of the full Riemann singular part, whereas at points where  $\Sigma$  is null this is not the case.

The above theorems seem to depend on the rigging vector  $\vec{\ell}$ , even though we have not chosen this vector field on the hypersurface. Therefore, it would be very interesting to see that the results do not depend, in fact, on the rigging vector  $\vec{\ell}$ . This can be established straightforwardly.

**Theorem 8** *The condition  $[\mathcal{H}_{\alpha\beta}] = 0$  is invariant under arbitrary changes of the rigging vector  $\vec{\ell}$ .*

*Proof:* Let us consider another rigging  $\vec{\ell}'$ . Its difference tensor  $[\mathcal{H}'_{\alpha\beta}]$  is, of course

$$[\mathcal{H}'_{\alpha\beta}] = \frac{1}{2} P'^{\mu}_{\alpha} P'^{\nu}_{\beta} \zeta_{\mu\nu}$$

where  $P'^{\mu}_{\alpha} = \delta^{\mu}_{\alpha} - \ell'^{\mu} n_{\alpha}$  is the projection tensor associated to the rigging  $\vec{\ell}'$ . Using now the decomposition (62) of  $\zeta_{\mu\nu}$  and the fact that  $P'^{\alpha}_{\mu}$  is a projector to the hypersurface in the sense that  $P'^{\alpha}_{\mu} n_{\alpha} = 0$ , we obtain

$$[\mathcal{H}'_{\alpha\beta}] = \frac{1}{2} P'^{\mu}_{\alpha} P'^{\nu}_{\beta} \zeta_{\mu\nu} = P'^{\mu}_{\alpha} P'^{\nu}_{\beta} [\mathcal{H}_{\mu\nu}]$$

This equation shows that if  $[\mathcal{H}_{\mu\nu}] = 0$  then  $[\mathcal{H}'_{\mu\nu}] = 0$ , and vice versa due to the symmetric role played by  $\vec{\ell}$  and  $\vec{\ell}'$  in this reasoning.

Let us study now what is the difference of the objects in the hypersurface, like  $\Psi_b^a$ ,  $\varphi_a$ ,  $\tilde{\Gamma}_{bc}^a$ ,  $\Gamma_{bc}^a$  or  $K_{ab}$ , when calculated from  $V^+$  or  $V^-$ . It is very easy from their definitions to find the following results

$$[\Psi_b^a] = \frac{1}{2} g^{ac} e_c^{\mu} e_b^{\nu} \zeta_{\mu\nu} = g^{ac} e_c^{\mu} e_b^{\nu} [\mathcal{H}_{\mu\nu}] = g^{ac} [\mathcal{H}_{cb}] \quad , \quad (72)$$

$$[\varphi_a] = \frac{1}{2} n^b e_b^{\mu} e_a^{\nu} \zeta_{\mu\nu} = n^b e_b^{\mu} e_a^{\nu} [\mathcal{H}_{\mu\nu}] = n^b [\mathcal{H}_{ba}] \quad , \quad (73)$$

$$[K_{ab}] = \frac{1}{2} (\vec{n} \cdot \vec{n}) e_a^{\mu} e_b^{\nu} \zeta_{\mu\nu} = (\vec{n} \cdot \vec{n}) e_a^{\mu} e_b^{\nu} [\mathcal{H}_{\mu\nu}] = (\vec{n} \cdot \vec{n}) [\mathcal{H}_{ab}] \quad , \quad (74)$$

$$[\Gamma_{bc}^a] = -\frac{1}{2} n^a e_b^{\mu} e_c^{\nu} \zeta_{\mu\nu} = -n^a e_b^{\mu} e_c^{\nu} [\mathcal{H}_{\mu\nu}] = -n^a [\mathcal{H}_{bc}] \quad , \quad (75)$$

$$[\tilde{\Gamma}_{bc}^a] = 0 \quad . \quad (76)$$

From (76) we learn that the rigged metric connection is *always*, and for any rigging, continuous. In this sense, the rigged metric connection is more intrinsic for general hypersurfaces than the rigged connection. Consequence of this, (or directly of equations (75) and (73)), is that  $\Gamma_c$  has always the same definitions from both sides of the hypersurface, as is trivial from its expression  $\Gamma_c = \partial_c \log(\eta_{123})$ . The rigged connection, as we can see from (75), is discontinuous in general unless either we have already matched properly such that the junction conditions hold or in the case of non-null hypersurfaces when the rigging is chosen canonically as the normal vector (in which case we have  $n^a = 0$ ).

With regard to the second fundamental form, from (74) we have the standard and very well-known result (see, for instance, [24],[21], [22]):

**Theorem 9** *At any point  $x \in \Sigma$  where the hypersurface is non-null, the necessary and sufficient condition for the singular part of the Riemann tensor to vanish is that the second fundamental form be continuous across the hypersurface.*

On the other hand, the second fundamental form is always continuous at any point  $x \in \Sigma$  where the hypersurface is null.

This theorem has its counterpart in the case of general hypersurfaces, but we must use the tensor  $\Psi_b^a$  instead of the second fundamental form, and also we have to choose a non-null rigging. The precise statement, which follows directly from equation (72) and proposition 4, is

**Theorem 10** *At any point  $x \in \Sigma$ , the necessary and sufficient condition for the singular part of the Riemann tensor to vanish is that the tensor  $\Psi_b^a$ , constructed with any non-null rigging, be the same defined from the region  $V^+$  and from the region  $V^-$ .*

Next, we are going to show that the second Bianchi identity holds in the distributional sense independently of whether or not the junction conditions are satisfied (of course, the preliminary junction conditions and a  $C^3$  differentiability for  $g^\pm$  are both assumed). To prove it, we do not have to extend any object in the hypersurface outside from it, but we need to know the covariant derivative of the Riemann tensor distribution. Although in the general case we cannot define the covariant derivative of arbitrary distributions because the connection symbols are discontinuous across the hypersurface, for the Riemann distribution this can be done without problems at once. This is due to the fact that the Riemann distribution components are able to act on the following set of test functions: those with a well-defined restriction to the hypersurface and such that they are *just* locally integrable both in the spacetime and in the hypersurface. From definition (109) we can find after some simple calculations

$$\begin{aligned} \langle \nabla_\mu \underline{R}^\alpha_{\beta\gamma\delta}, Y \rangle &= \int_{V^+} \nabla_\mu R^{+\alpha}_{\beta\gamma\delta} Y \boldsymbol{\eta} + \int_{V^-} \nabla_\mu R^{-\alpha}_{\beta\gamma\delta} Y \boldsymbol{\eta} + \int_\Sigma n_\mu [R^\alpha_{\beta\gamma\delta}] Y d\sigma + \\ &+ \int_\Sigma \left\{ \left( -H^\alpha_{\sigma\gamma\delta} \Gamma^\sigma_{\beta\mu} - H^\alpha_{\beta\sigma\delta} \Gamma^\sigma_{\gamma\mu} - H^\alpha_{\beta\gamma\sigma} \Gamma^\sigma_{\delta\mu} + H^\nu_{\beta\gamma\delta} \Gamma^\alpha_{\nu\mu} - H^\alpha_{\beta\gamma\delta} \Gamma^\rho_{\mu\rho} \right) Y - H^\alpha_{\beta\gamma\delta} \partial_\mu Y \right\} d\sigma. \end{aligned} \quad (77)$$

From here it follows that we must know the discontinuity of the Riemann tensor across the hypersurface. Equation (50) applied to the Christoffel symbols gives

$$[\partial_\gamma \Gamma^\alpha_{\beta\delta}] = n_\gamma A^\alpha_{\beta\delta} + \omega_\gamma^a \partial_a [\Gamma^\alpha_{\beta\delta}] \quad ,$$

while the discontinuity  $[\Gamma^\alpha_{\gamma\rho} \Gamma^\rho_{\beta\delta}]$  is easily found to be

$$[\Gamma^\alpha_{\gamma\rho} \Gamma^\rho_{\beta\delta}] = \Gamma^\alpha_{\gamma\rho} \Big|_\Sigma [\Gamma^\rho_{\beta\delta}] + [\Gamma^\alpha_{\gamma\rho}] \Gamma^\rho_{\beta\delta} \Big|_\Sigma \quad .$$

Thus, we explicitly have (77) in terms of known objects and we can evaluate

$$\nabla_\mu \underline{R}^\alpha_{\beta\gamma\delta} + \nabla_\gamma \underline{R}^\alpha_{\beta\delta\mu} + \nabla_\delta \underline{R}^\alpha_{\beta\mu\gamma} \quad . \quad (78)$$

After a long but straightforward calculation involving a decomposition of the partial derivative of the test function  $\partial_\mu Y = n_\mu \ell^\nu \partial_\nu Y + \omega_\mu^a \partial_a Y$ , a careful integration by parts and using the fact that

$$n_\mu H_{\beta\gamma\delta}^\alpha + n_\gamma H_{\beta\delta\mu}^\alpha + n_\delta H_{\beta\mu\gamma}^\alpha = 0$$

as follows from the explicit expression (54) of  $H_{\beta\gamma\delta}^\alpha$ , together with the Bianchi identities for the Riemann tensors in  $V^+$  and  $V^-$ , we find that expression (78) applied to a test function  $Y$  produces the following result

$$\int_\Sigma \left\{ \left[ \Gamma_{\beta\delta}^\alpha \right] \left( \omega_\mu^a \partial_a n_\gamma - \omega_\gamma^a \partial_a n_\mu + n_\gamma (\partial_a \omega_\mu^a + \omega_\mu^a \Gamma_a) - n_\mu (\partial_a \omega_\gamma^a + \omega_\gamma^a \Gamma_a) \right) - \Gamma_{\mu\nu}^\nu H_{\beta\gamma\delta}^\alpha + \text{c.t.} \right\} d\sigma$$

where c.t. represents the two other terms obtained from the one shown by permuting cyclically the indexes  $\mu$ ,  $\gamma$  and  $\delta$ . Taking into account the formulas of previous sections, it is not difficult to see that

$$\omega_\mu^a \partial_a n_\gamma - \omega_\gamma^a \partial_a n_\mu + n_\gamma (\partial_a \omega_\mu^a + \omega_\mu^a \Gamma_a) - n_\mu (\partial_a \omega_\gamma^a + \omega_\gamma^a \Gamma_a) = n_\gamma \Gamma_{\mu\nu}^\nu - n_\mu \Gamma_{\gamma\nu}^\nu$$

and recalling again expression (54) we arrive then to the Bianchi identities

$$\nabla_\mu \underline{R}_{\beta\gamma\delta}^\alpha + \nabla_\gamma \underline{R}_{\beta\delta\mu}^\alpha + \nabla_\delta \underline{R}_{\beta\mu\gamma}^\alpha = 0. \quad (79)$$

Of course, from this we also have

$$\nabla_\mu \underline{G}^{\mu\nu} = 0 \quad ,$$

so that, when the metric is continuous, the energy-momentum tensor distribution is conserved in the distributional sense.

## 6 Physical Implications of the Junction Conditions.

Let us now assume that we have made a proper matching between spacetimes such that the singular part of the Riemann tensor distribution vanishes or, equivalently, for every rigging vector field  $\vec{\ell}$  we have  $[\mathcal{H}_{\mu\nu}] = 0$ . Our aim is to find the allowable discontinuities of the Riemann tensor, that is to say, we want to know which physical components of the curvature, i.e. of the Einstein tensor (matter contents) and the Weyl tensor (pure gravitational field), can have discontinuities across the hypersurface. Given that  $[\mathcal{H}_{\mu\nu}] = 0$ , we get from formulas (72-75) above that

$$[\Psi_b^a] = 0, \quad [\varphi_c] = 0, \quad [K_{ab}] = 0, \quad [\Gamma_{bc}^a] = 0$$

and therefore the covariant derivative  $\bar{\nabla}$  in the hypersurface has the same definition coming from  $V^+$  or  $V^-$ . Using now the Gauss equation (12) and the three Codazzi

equations (13,15,16) we immediately find

$$\begin{aligned} [\omega_\alpha^d R_{\beta\gamma\delta}^\alpha e_a^\beta e_b^\gamma e_c^\delta] &= 0, & [n_\mu R_{\alpha\beta\gamma}^\mu e_a^\alpha e_b^\beta e_c^\gamma] &= 0, \\ [\omega_\mu^c R_{\alpha\beta\gamma}^\mu \ell^\alpha e_a^\beta e_b^\gamma] &= 0, & [n_\mu R_{\alpha\beta\gamma}^\mu \ell^\alpha e_a^\beta e_b^\gamma] &= 0. \end{aligned}$$

Noting that  $\{\mathbf{n}, \omega^c\}$  and  $\{\vec{\ell}, \vec{e}_a\}$  are bases of their respective tangent spaces, we have

**Theorem 11** *If the junction conditions  $[\mathcal{H}_{\mu\nu}] = 0$  are satisfied, then*

$$[R_{\alpha\beta\gamma}^\mu] e_a^\beta e_b^\gamma = 0. \quad (80)$$

*These are fourteen independent relations so that fourteen out of the twenty components of the Riemann tensor must be continuous. Thus, only six independent discontinuities in the curvature are allowed.*

We can rewrite equations (80) in several different forms. First of all, given three independent vectors  $\vec{e}_a$  and their normal one-form  $\mathbf{n}$ , it is a general result for an object ( $V$ ) with two covariant antisymmetric indexes that

$$(V)_{\lambda\mu} e_a^\lambda e_b^\mu = 0 \Leftrightarrow n_\sigma (V)_{\lambda\mu} + n_\lambda (V)_{\mu\sigma} + n_\mu (V)_{\sigma\lambda} = 0$$

where we have written ( $V$ ) because this object can have more than two indexes. Therefore the continuity conditions for the Riemann tensor can be rewritten as

$$n_\sigma [R_{\beta\lambda\mu}^\alpha] + n_\lambda [R_{\beta\mu\sigma}^\alpha] + n_\mu [R_{\beta\sigma\lambda}^\alpha] = 0. \quad (81)$$

It is a known result that (81) is still equivalent to the existence of a symmetric two-covariant tensor  $B_{\mu\nu}$  such that

$$[R_{\alpha\beta\lambda\mu}] = n_\alpha n_\lambda B_{\beta\mu} - n_\lambda n_\beta B_{\alpha\mu} - n_\mu n_\alpha B_{\beta\lambda} + n_\mu n_\beta B_{\alpha\lambda}, \quad (82)$$

where  $B_{\lambda\mu}$  is a tensor field defined at the hypersurface and unique up to a transformation of the type

$$B'_{\lambda\mu} = B_{\lambda\mu} + X_\lambda n_\mu + n_\lambda X_\mu$$

with  $X_\lambda$  an arbitrary one-form. The ten independent components of  $B_{\lambda\mu}$  minus the four gauge freedoms  $X_\lambda$  give the six arbitrary possible discontinuities of the Riemann tensor. Direct consequences of (81) are the following

$$n_\sigma [R_{\beta\lambda\mu}^\sigma] = n_\lambda [R_{\beta\mu}] - n_\mu [R_{\beta\lambda}], \quad (83)$$

$$n_\sigma [R_{\beta\lambda\mu}^\sigma] e_a^\lambda = -n_\mu [R_{\beta\lambda}] e_a^\lambda \quad (84)$$

where we have not lost any information. In other words, equations (80), (81), (82), (83) and (84) are, all of them, equivalent to each other.

We are now ready to study which of the six allowable curvature discontinuities are matter discontinuities or pure gravitational (Weyl) ones. Contracting indexes  $\beta$  and  $\mu$  in equation (83) we get

$$n^\sigma [R_{\sigma\lambda}] = \frac{1}{2} n_\lambda [R] \quad (85)$$

or, what is the same,

$$n^\sigma [G_{\sigma\lambda}] = 0 \quad (86)$$

where, of course,  $G_{\sigma\lambda}$  is the Einstein tensor of the manifold. These relations were known in the case of non-null junction hypersurfaces as the Israel conditions [24]. This equation tells us that four components of the Einstein tensor cannot have any discontinuities across the junction hypersurface. In order to see which are the remaining ten continuous components of the curvature, we use the decomposition of the Riemann tensor in terms of the Weyl tensor, Ricci tensor and scalar curvature

$$R_{\beta\lambda\mu}^\alpha = C_{\beta\lambda\mu}^\alpha + \frac{1}{2} (R_{\lambda\alpha}^\beta g_{\beta\mu} - R_{\mu\alpha}^\beta g_{\beta\lambda} + \delta_\lambda^\alpha R_{\beta\mu} - \delta_\mu^\alpha R_{\beta\lambda}) - \frac{1}{6} R (\delta_\lambda^\alpha g_{\beta\mu} - \delta_\mu^\alpha g_{\beta\lambda}) \quad (87)$$

and then we can rewrite equations (83) and (84), respectively, in the the following form

$$n_\sigma [C_{\beta\lambda\mu}^\sigma] = \frac{1}{12} [R] (g_{\beta\lambda} n_\mu - g_{\beta\mu} n_\lambda) + \frac{1}{2} (n_\lambda [R_{\beta\mu}] - n_\mu [R_{\beta\lambda}]) \quad , \quad (88)$$

$$n_\sigma [C_{\beta\lambda\mu}^\sigma] e_a^\lambda = \frac{1}{12} n_\mu e_a^\lambda ([R] g_{\beta\lambda} - 6 [R_{\beta\lambda}]) \quad (89)$$

where we have used (85). Despite of this, these two relations are again fully equivalent to the general condition (80). From the last equation it is immediate that

$$n_\sigma [C_{\beta\lambda\mu}^\sigma] e_a^\lambda e_b^\mu = 0 \quad (90)$$

and this equation contains five independent relations so that these five out of the ten components of the Weyl tensor must be continuous across the hypersurface. We have hitherto decomposed the fourteen conditions on the Riemann tensor into four on the Einstein tensor (or Ricci tensor) and five on the Weyl tensor. We will complete these relations by identifying the possible discontinuities of the Riemann tensor and writing down all the components of the Ricci and Weyl tensors in terms of these discontinuities. From formula (82) it is clear that the independent discontinuities of the Riemann tensor are

$$\ell^\alpha e_a^\beta \ell^\lambda e_b^\mu [R_{\alpha\beta\lambda\mu}] = e_a^\beta e_b^\mu B_{\beta\mu} \equiv B_{ab}, \quad (91)$$

where obviously  $B_{ab}$  are independent of the gauge freedom of  $B_{\alpha\beta}$  and are the six independent allowed discontinuities in the Riemann tensor. The discontinuities of the Ricci and Weyl tensor can be straightforwardly written in terms of these quantities using equation (82) and the decomposition (87) as

$$[R] = 2(\vec{n} \cdot \vec{n}) g^{ab} B_{ab} - 2n^a n^b B_{ab} \quad (92)$$

$$\ell^\alpha \ell^\beta [R_{\alpha\beta}] = g^{ab} B_{ab} \quad (93)$$

$$\ell^\alpha e_b^\beta [R_{\alpha\beta}] = -n^a B_{ab} \quad (94)$$

$$e_a^\alpha e_b^\beta [R_{\alpha\beta}] = (\vec{n} \cdot \vec{n}) B_{ab} \quad (95)$$

$$\begin{aligned} \ell^\alpha e_a^\beta \ell^\lambda e_b^\mu [C_{\alpha\beta\lambda\mu}] &= \frac{2 - (\vec{n} \cdot \vec{n}) (\vec{\ell} \cdot \vec{\ell})}{2} B_{ab} - \frac{1}{2} n^c (B_{cb} \ell_a + B_{ca} \ell_b) - \frac{\ell_a \ell_b}{3} \left( (\vec{n} \cdot \vec{n}) g^{cd} B_{cd} \right. \\ &\quad \left. - n^c n^d B_{cd} \right) - \bar{g}_{ab} \left\{ \frac{\vec{\ell} \cdot \vec{\ell}}{3} n^c n^d B_{cd} + \frac{3 - 2(\vec{n} \cdot \vec{n}) (\vec{\ell} \cdot \vec{\ell})}{6} g^{cd} B_{cd} \right\} \end{aligned} \quad (96)$$

$$\begin{aligned} \ell^\alpha e_b^\beta e_c^\lambda e_d^\mu [C_{\alpha\beta\lambda\mu}] &= \frac{1}{2} n^a (B_{ac} \bar{g}_{bd} - B_{ad} \bar{g}_{bc}) + \frac{\vec{n} \cdot \vec{n}}{2} (B_{bc} \ell_d - B_{bd} \ell_c) \\ &\quad + \frac{1}{3} \left( (\vec{n} \cdot \vec{n}) g^{ef} B_{ef} - n^e n^f B_{ef} \right) (\bar{g}_{bd} \ell_c - \bar{g}_{bc} \ell_d) \end{aligned} \quad (97)$$

$$\begin{aligned} e_a^\alpha e_b^\beta e_c^\lambda e_d^\mu [C_{\alpha\beta\lambda\mu}] &= \frac{\vec{n} \cdot \vec{n}}{2} (B_{ad} \bar{g}_{bc} - B_{ac} \bar{g}_{bd} + B_{bc} \bar{g}_{ad} - B_{bd} \bar{g}_{ac}) \\ &\quad + \frac{1}{3} \left( (\vec{n} \cdot \vec{n}) g^{ef} B_{ef} - n^e n^f B_{ef} \right) (\bar{g}_{ac} \bar{g}_{bd} - \bar{g}_{ad} \bar{g}_{bc}). \end{aligned} \quad (98)$$

These equations contain both the Israel conditions (86) and the relations concerning only the Weyl tensor (90) and include all the the information about the continuities of the Riemann tensor. The question now is to rewrite these equations in terms of six components of the Weyl or Ricci tensor which can be arbitrarily discontinuous across the hypersurface. For an everywhere non-degenerate hypersurface these components can be chosen as the tangent part of the Ricci or Einstein tensor, as is clear from relation (95), but for a general hypersurface those are not arbitrary because they must tend to zero in the singular points where the hypersurface is degenerate. A suitable set of six independent components whose discontinuities are arbitrary everywhere in a general hypersurface is

$$\ell^\alpha \ell^\beta [R_{\alpha\beta}] - \frac{\vec{\ell} \cdot \vec{\ell}}{3} [R] \equiv \Omega \quad , \quad \ell^\alpha e_a^\beta \ell^\lambda e_b^\mu [C_{\alpha\beta\lambda\mu}] \equiv S_{ab}. \quad (99)$$

The six components  $\ell^\alpha e_b^\beta \ell^\lambda e_c^\mu [C_{\alpha\beta\lambda\mu}]$  satisfy the relation

$$g^{ab} S_{ab} \equiv g^{ab} \ell^\alpha e_b^\beta \ell^\lambda e_c^\mu [C_{\alpha\beta\lambda\mu}] = 0,$$

consequence of the fact that the Weyl tensor is traceless. So the components written above constitute really a set of six independent quantities.



From equation (96) it is clear that  $B_{ab}$  can be written in terms of these components when  $(\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell}) \neq 2$ .

$$B_{ab} = \frac{2}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \left( S_{ab} + n^c S_{cb} \ell_a + n^c S_{ca} \ell_b + \frac{1}{2} \bar{g}_{ab} \Omega \right). \quad (100)$$

This is not a restriction because this is a condition only on the rigging but not on the point type of the hypersurface. We can always choose a rigging vector  $\vec{\ell}$  that satisfies the above inequality. Thus, in terms of the quantities (99) the fourteen continuity relations are

$$[R] = \frac{6}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \left( -2n^e n^d S_{ed} + (\vec{n} \cdot \vec{n}) \Omega \right) \quad (101)$$

$$\ell^\alpha \ell^\beta [R_{\alpha\beta}] = \frac{(2 + (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})) \Omega - 4(\vec{\ell} \cdot \vec{\ell}) n^e n^d S_{ed}}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \quad (102)$$

$$\ell^\alpha e_c^\beta [R_{\alpha\beta}] = -2n^b S_{bc} + \frac{\ell_c}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \left( (\vec{n} \cdot \vec{n}) \Omega - 2n^e n^d S_{ed} \right) \quad (103)$$

$$e_a^\alpha e_b^\beta [R_{\alpha\beta}] = \frac{\vec{n} \cdot \vec{n}}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} (2S_{ab} + 2(n^e S_{eb} \ell_a + n^e S_{ea} \ell_b) + \bar{g}_{ab} \Omega) \quad (104)$$

$$\begin{aligned} \ell^\alpha e_a^\beta e_b^\lambda e_c^\mu [C_{\alpha\beta\lambda\mu}] &= \left( \bar{g}_{ac} n^d S_{bd} - \bar{g}_{ab} n^d S_{dc} \right) - \frac{n^d n^e S_{de} (\bar{g}_{ac} \ell_b - \bar{g}_{ab} \ell_c)}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \\ &+ \frac{(\vec{n} \cdot \vec{n}) \left\{ \ell_a (\ell_c n^d S_{db} - \ell_b n^d S_{dc}) - (\ell_b S_{ac} - \ell_c S_{ab}) \right\}}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \end{aligned} \quad (105)$$

$$\begin{aligned} e_a^\alpha e_b^\beta e_c^\lambda e_d^\mu [C_{\alpha\beta\lambda\mu}] &= \frac{\vec{n} \cdot \vec{n}}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} \left\{ \bar{g}_{bc} S_{ad} - \bar{g}_{bd} S_{ac} + \bar{g}_{ad} S_{bc} - \bar{g}_{ac} S_{bd} \right. \\ &+ n^e S_{ea} (\ell_d \bar{g}_{bc} - \ell_c \bar{g}_{bd}) + n^e S_{eb} (\ell_c \bar{g}_{ad} - \ell_d \bar{g}_{ac}) \\ &+ n^e S_{ed} (\ell_a \bar{g}_{bc} - \ell_b \bar{g}_{ac}) + n^e S_{ec} (\ell_b \bar{g}_{ad} - \ell_a \bar{g}_{bc}) \left. \right\} \\ &+ \frac{2n^e n^f S_{ef}}{2 - (\vec{n} \cdot \vec{n})(\vec{\ell} \cdot \vec{\ell})} (\bar{g}_{ad} \bar{g}_{bc} - \bar{g}_{ac} \bar{g}_{bd}). \end{aligned} \quad (106)$$

Several interesting considerations can be made at the sight of these equations about the behaviour of the discontinuities of the Ricci and Weyl tensors across the junction hypersurface of two properly matched spacetimes, but as they are self-evident from the equations we will not discuss them unless for the most important ones. We write them down in the form of

**Theorem 12** *If the junction conditions are verified on the matching hypersurface, the following properties about the discontinuities of the Riemann tensor hold:*

1. *The four normal components (86) of the energy-momentum tensor are necessarily continuous across the hypersurface (Israel conditions).*
2. *The five components (90) of the Weyl tensor are continuous across the matching hypersurface.*
3. *A suitable set of six independent allowable discontinuities of the Ricci and Weyl tensors for general hypersurfaces which change in character is given by (99).*
4. *At any point  $x \in \Sigma$  where the matching hypersurface is non-null the six independent allowable discontinuities can be chosen as arbitrary matter discontinuities given by  $[T_{\mu\nu}e_a^\mu e_b^\nu]$ .*
5. *For null points the tangent part of the Ricci tensor must be continuous (see (95)) across the hypersurface. Moreover, four of the six independent allowable discontinuities can be chosen as matter discontinuities given by  $[\ell^\mu R_{\mu\nu}]$  and the other two arbitrary discontinuities are given by the spinorial component  $[\Psi_4]$  in any null tetrad with  $\vec{n}$  as the first real null vector.*
6. *Near the singular points of the hypersurface the tangent components of the Ricci tensor must tend to zero at least as the first power of the norm of the normal vector (see (95)).*

To end this section we consider the question if there exists any differential equation governing the evolution of the arbitrary discontinuities of the Riemann tensor. The Bianchi identities, which are true both in  $V^+$  and  $V^-$  imply obviously that

$$[\nabla_\rho R_{\alpha\beta\lambda\mu} + \nabla_\lambda R_{\alpha\beta\mu\rho} + \nabla_\mu R_{\alpha\beta\rho\lambda}] = 0. \quad (107)$$

Due to the general formula (50), the discontinuity of the covariant derivative of the Riemann tensor can be written

$$[\nabla_\rho R_{\alpha\beta\lambda\mu}] = n_\rho t_{\alpha\beta\lambda\mu} + \omega_\rho^a (e_a^\sigma \nabla_\sigma [R_{\alpha\beta\lambda\mu}]),$$

where  $t_{\alpha\beta\lambda\mu}$  is an arbitrary tensor on the hypersurface with the symmetries of a Riemann tensor. Substituting in the second term of this relation the equation for the discontinuity of the Riemann tensor (82) and using some of the relations written down in a previous section it can be proven that the Bianchi discontinuity relation (107) is completely equivalent to the relation

$$\begin{aligned} t_{\alpha\beta\lambda\mu} e_b^\lambda e_d^\mu &= \frac{1}{2} (K_{ad} B_{bc} - K_{cd} B_{ba} - K_{ab} B_{dc} + K_{cb} B_{da}) (\omega_\alpha^a \omega_\beta^c - \omega_\beta^a \omega_\alpha^c) \\ &+ (\bar{\nabla}_b B_{ad} - \bar{\nabla}_d B_{ab} - 2\varphi_b B_{ad} + 2\varphi_d B_{ab}) (n_\alpha \omega_\beta^a - n_\beta \omega_\alpha^a). \end{aligned} \quad (108)$$

As before, these are fourteen relations for the twenty independent components of  $t_{\alpha\beta\lambda\mu}$ , so there appear six new arbitrary independent discontinuities in  $\nabla_\rho R_{\alpha\beta\lambda\mu}$ . The other six relations contained in the Bianchi identities are identically verified for the derivatives of the discontinuity of the Riemann tensor and do not involve the tensor  $t_{\alpha\beta\lambda\mu}$ . Thus the Bianchi discontinuity relations do not provide us with any evolution equation for the discontinuity of the Riemann tensor. However, we see from (108) that the discontinuities in the first derivatives of the Riemann tensor (apart from the six new arbitrary ones) involve not only the discontinuities  $B_{ab}$  of the Riemann tensor itself and its first derivatives, but also intrinsic properties of the matching hypersurface, namely, the second fundamental form and the one-form  $\varphi$ .

## 7 Acknowledgements

The present work has been partially supported by DGICYT project PB90-0482-C02-01. M.M. wishes to thank the Direcci General d'Universitats, Generalitat de Catalunya for financial support.

## 8 Appendix.

In order to define tensor distributions (see, for instance [20],[16]) it is necessary to construct the set  $\mathcal{D}(V_4)$  of test tensor fields, that is, the set of  $C^\infty$  tensor fields of any order with compact support in the manifold. We denote by  $\mathcal{D}_p^q$  the subset of p-covariant q-contravariant tensor fields with compact support. Then, p-covariant q-contravariant tensor distributions  $\chi_p^q$  are defined as linear and continuous functionals from  $\mathcal{D}_q^p$  to the real numbers, that is to say

$$\begin{aligned} \chi_p^q : \mathcal{D}_q^p &\rightarrow \mathbb{R} \\ Y_q^p &\rightarrow \chi_p^q(Y_q^p) \equiv \langle \chi_p^q, Y_q^p \rangle . \end{aligned}$$

The sum of two tensor distributions of the same type and the product of a tensor distribution by a real number can be defined in the usual way. With these definitions the space of tensor distributions of a given type constitutes a vector space. Given any locally integrable p-covariant q-contravariant tensor field  $T_p^q$  in an oriented manifold, we can define a p-covariant q-contravariant tensor distribution associated to it as follows

$$\begin{aligned} \underline{T}_p^q : \mathcal{D}_q^p &\rightarrow \mathbb{R} \\ Y_q^p &\rightarrow \langle \underline{T}_p^q, Y_q^p \rangle \equiv \int_{V_4} (T, Y) \boldsymbol{\eta} \end{aligned}$$

where  $(T, Y)$  means tensor contraction on all indexes in the natural order. As  $Y$  is in  $\mathcal{D}_q^p$ , its contraction with  $T$  is a locally integrable scalar function with compact support

in the manifold and therefore the above integral is well-defined. We will repeatedly use the convention of distinguishing between a tensor field and the distribution it defines by using an underline as before. As is seen in the last formula the tensor distribution associated to a tensor field can act not only on  $C^\infty$  tensor fields with compact support but also on continuous ones. We will consider the action of a tensor distribution always that this action can be defined.

The components of a tensor distribution  $\chi$  in a coordinate system are scalar distributions  $\chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}$  defined by

$$\langle \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}, Y \rangle \equiv \left\langle \chi_p^q, Y dx^{\alpha_1} \otimes \dots \otimes dx^{\alpha_p} \otimes \frac{\partial}{\partial x^{\beta_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{\beta_q}} \right\rangle$$

where  $Y$  is a function with compact support. With this definition it is not difficult to prove the following expression

$$\langle \chi_p^q, Y_q^p \rangle = \langle \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}, Y_{\alpha_1 \dots \alpha_q}^{\beta_1 \dots \beta_p} \rangle .$$

The contraction of some indexes of a tensor distribution can be defined as the tensor distribution whose components in a coordinate system are those obtained by contracting the desired indexes of its components. This definition is seen to be well defined and independent of the coordinate system. Moreover, the tensor product of a tensor distribution  $\chi_p^q$  by a tensor field  $T_s^r$  can be defined, (in general, only when this tensor field is  $C^\infty$  but, as we have already discussed, also in more general cases sometimes), as the  $(p+s)$ -covariant  $(q+r)$ -contravariant tensor distribution acting as follows

$$\langle T_s^r \otimes \chi_p^q, Y_{r+q}^{s+p} \rangle \equiv \langle \chi_p^q, (T, Y)_q^p \rangle$$

where  $(T, Y)_q^p$  is the element of  $\mathcal{D}_q^p$  obtained by contracting  $T$  with the first indexes of  $Y$  in order.

We are now going to define covariant derivatives of tensor distributions. To that aim, we have to consider Riemannian manifolds (or at least with a linear connection satisfying Gauss theorem) such that the Christoffe, symbols are, in principle,  $C^\infty$  in each coordinate system. The definition of covariant derivative of a tensor distribution generalizes the concept of covariant derivative of tensor fields in the sense that for tensor distributions coming from a tensor field it gives the tensor distribution  $C^\infty$  tensor fields of any order with compact support in the manifold. We denote by  $\mathcal{D}_p^q$  the subset of  $p$ -covariant  $q$ -contravariant tensor fields with compact support. Then,  $p$ -covariant  $q$ -contravariant tensor distributions  $\chi_p^q$  are defined as linear and continuous functionals from  $\mathcal{D}_q^p$  to the real numbers, that is to say

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where  $(T, Y)$  means tensor contraction on all indexes in the natural order. As  $Y$  is in  $\mathcal{D}_q^p$ , its contraction with  $T$  is a locally integrable scalar function with compact support in the manifold and therefore the above integral is well-defined. We will repeatedly use the convention of distinguishing between a tensor field and the distribution it defines by using an underline as before. As is seen in the last formula the tensor distribution associated to a tensor field can act not only on  $C^\infty$  tensor fields with compact support but also on continuous ones. We will consider the action of a tensor distribution always that this action can be defined.

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where  $Y$  is a function with compact support. With this definition it is not difficult to prove the following expression

$$\langle \chi_p^q, Y_q^p \rangle = \langle \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}, Y_{\alpha_1 \dots \alpha_q}^{\beta_1 \dots \beta_p} \rangle .$$

The contraction of some indexes of a tensor distribution can be defined as the tensor distribution whose components in a coordinate system are those obtained by contracting the desired indexes of its components. This definition is seen to be well defined and independent of the coordinate system. Moreover, the tensor product of a tensor distribution  $\chi_p^q$  by a tensor field  $T_s^r$  can be defined, (in general, only when this tensor field is  $C^\infty$  but, as we have already discussed, also in more general cases sometimes), as the (p+s)-covariant (q+r)-contravariant tensor distribution acting as follows

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We are now going to define covariant derivatives of tensor distributions. To that aim, we have to consider Riemannian manifolds (or at least with a linear connection

satisfying Gauss theorem) such that the Christoffel symbols are, in principle,  $C^\infty$  in each coordinate system. The definition of covariant derivative of a tensor distribution generalizes the concept of covariant derivative of tensor fields in the sense that for tensor distributions coming from a tensor field it gives the tensor distribution associated to the usual covariant derivative of the tensor field. This definition is

$$\langle \nabla \chi_p^q, Y_q^{p+1} \rangle \equiv - \langle \chi_p^q, (DY)_q^p \rangle$$

where  $(DY)_{\beta_1 \dots \beta_q}^{\alpha_1 \dots \alpha_p} = \nabla_\mu Y_{\beta_1 \dots \beta_q}^{\mu \alpha_1 \dots \alpha_p}$ . With this definition, the components of the covariant derivative are the scalar distributions acting as

$$\begin{aligned} \langle \nabla_\mu \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}, Y \rangle &= - \langle \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_q}, \partial_\mu Y + \Gamma_{\rho\mu}^\rho Y \rangle - \sum_{i=1}^p \langle \chi_{\beta_1 \dots \beta_{i-1} \rho \beta_{i+1} \dots \beta_p}^{\alpha_1 \dots \alpha_q}, \Gamma_{\beta_i \mu}^\rho Y \rangle \\ &\quad + \sum_{j=1}^q \langle \chi_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_{j-1} \rho \alpha_{j+1} \dots \alpha_q}, \Gamma_{\rho\mu}^{\alpha_j} Y \rangle. \end{aligned} \quad (109)$$

In the case of a scalar distribution  $\chi$  we have therefore

$$\langle \nabla_\mu \chi, Y \rangle \equiv \langle \partial_\mu \chi, Y \rangle = - \langle \chi, \partial_\mu Y + \Gamma_{\rho\mu}^\rho Y \rangle$$

The last relations are written in the case of a Riemannian (or at least linear) manifold but they also hold in the case of n-dimensional manifolds possessing a  $C^1$  volume form  $\eta$  by substituting  $\Gamma_{\rho\mu}^\rho$  for  $\Gamma_\alpha$  defined as

$$\partial_\mu \eta_{\beta_1 \dots \beta_n} \equiv \Gamma_\mu \eta_{\beta_1 \dots \beta_n} .$$

From now on, we will restrict ourselves to the case studied in chapter 3 whenever the preliminary junction conditions are satisfied, i.e. a spacetime  $V_4$  with a hypersurface  $\Sigma$  such that the metric tensor is continuous but not differentiable across this hypersurface. Thus, the Christoffel symbols do not exist at points of  $\Sigma$  but they do everywhere outside this hypersurface. First of all we will define the so-called step Heaviside function of  $\Sigma$  by

$$\begin{aligned} \theta : V_4 &\rightarrow \mathbb{R} \\ \theta &= \begin{cases} 1 & \text{in } V^+ \\ a & \text{in } \Sigma \\ 0 & \text{in } V^- \end{cases} \end{aligned} \quad (110)$$

where  $a$  is an arbitrary real number. This function is locally integrable and therefore it defines a scalar distribution  $\underline{\theta}$  in the natural way:

$$\langle \underline{\theta}, Y \rangle = \int_{V^+} Y \eta$$

The distribution  $\underline{\theta}$  does not depend on the value of the function  $\theta$  on the hypersurface, and therefore we will not fix this value at the moment.

In the case under consideration, the object  $\Gamma_\mu$  given above does not exist at the hypersurface but, in fact, this is not an obstacle to define the partial derivative of some scalar distributions. In particular, let us consider a scalar function  $f$  which is discontinuous across  $\Sigma$  but differentiable in  $V^+$  and  $V^-$ , and such that  $f$  and its first derivatives have definite limits on  $\Sigma$  coming from both  $V^+$  and  $V^-$ . If we call  $f^+$  its restriction to  $V^+$  and analogously for  $f^-$ , it is trivial to see that the distribution associated to that function, which exists because  $f$  is locally integrable, is

$$\underline{f} = f^+ \cdot \underline{\theta} + f^- \cdot (\underline{1} - \underline{\theta}) \ .$$

The partial derivative of this scalar distribution exists despite the discontinuity of  $\Gamma_\mu$  across  $\Sigma$ . Integrating by parts in  $V^+$  and  $V^-$  we find

$$\langle \nabla \underline{f}, \vec{Y} \rangle = \int_{V^+} Y^\mu \partial_\mu f^+ \boldsymbol{\eta} + \int_{V^-} Y^\mu \partial_\mu f^- \boldsymbol{\eta} + \int_\Sigma [f] Y^\mu d\sigma_\mu \quad (111)$$

where  $d\sigma_\mu$  is oriented from  $V^-$  towards  $V^+$  and  $[f]$  is a scalar function defined on  $\Sigma$ , called step of  $f$  at  $\Sigma$ , and defined by

$$\forall q \in \Sigma \quad [f](q) \equiv \lim_{x \xrightarrow{V^+} q} f^+(x) - \lim_{x \xrightarrow{V^-} q} f^-(x) \ .$$

We can rewrite all this by defining a one-covariant distribution  $\boldsymbol{\delta}$  as

$$\langle \boldsymbol{\delta}, \vec{Y} \rangle \equiv \int_\Sigma Y^\mu d\sigma_\mu = \int_\Sigma Y^\mu n_\mu d\sigma \ .$$

From equation (111), choosing  $f^+ = 1$  and  $f^- = 0$  it is direct that  $\boldsymbol{\delta}$  has an intrinsic definition as  $\boldsymbol{\delta} = \nabla \underline{\theta}$ . This distribution can act on every vector field defined at least at the points of the hypersurface which is locally integrable there. We can also define a scalar distribution  $\delta$  as follows

$$\langle \delta, Y \rangle \equiv \int_\Sigma Y d\sigma \ .$$

Of course,  $\delta$  depends on the choice of the normal form  $\mathbf{n}$  and we have  $\boldsymbol{\delta} = \mathbf{n} \cdot \delta$  or, in components,  $\delta_\mu = n_\mu \cdot \delta$ . Therefore equation (111) can be written as

$$\partial_\mu \underline{f} = \partial_\mu f^+ \cdot \underline{\theta} + \partial_\mu f^- \cdot (\underline{1} - \underline{\theta}) + [f] \cdot \delta_\mu \ . \quad (112)$$

We are now going to define the connection and the Riemann tensor in the manifold.  $g$  being a continuous tensor across the hypersurface we can write it as

$$g = (1 - \theta) g^- + \theta g^+$$

independently of the value of  $a$ , and the tensor distribution associated to it obviously is

$$\underline{g} = (\underline{1} - \underline{\theta}) \cdot g^- + \underline{\theta} \cdot g^+ .$$

As usual, we can define the connection symbols associated to the metric in the manifold. We denote by  $\Gamma^{+\alpha}_{\beta\gamma}$  the Christoffel symbols associated with  $g^+$  and defined on  $V^+ \cup \Sigma$ , by  $\Gamma^{-\alpha}_{\beta\gamma}$  those associated to  $g^-$  and defined on  $V^- \cup \Sigma$ , and finally by  $\underline{\Gamma}^{\alpha}_{\beta\gamma}$  the connection symbols associated with the whole metric distribution  $\underline{g}$ . The relation between them is obviously

$$\underline{\Gamma}^{\alpha}_{\beta\gamma} = (\underline{1} - \underline{\theta}) \cdot \Gamma^{-\alpha}_{\beta\gamma} + \underline{\theta} \cdot \Gamma^{+\alpha}_{\beta\gamma} . \quad (113)$$

However, in order to be able to define covariant derivatives of tensor distributions, and as is immediate from formula (109), we need to know the connection symbols not only as distributions but as functions in the manifold as well. We choose these functions in a natural way from the expression of the connection symbols as distributions. In consequence we have

$$\Gamma^{\alpha}_{\beta\gamma} = (1 - \theta) \Gamma^{-\alpha}_{\beta\gamma} + \theta \Gamma^{+\alpha}_{\beta\gamma} . \quad (114)$$

Using the definition of the Riemann tensor  $R^{\alpha}_{\beta\lambda\mu} = \partial_{\lambda}\Gamma^{\alpha}_{\beta\mu} - \partial_{\mu}\Gamma^{\alpha}_{\beta\lambda} + \Gamma^{\alpha}_{\lambda\rho}\Gamma^{\rho}_{\beta\mu} - \Gamma^{\alpha}_{\mu\rho}\Gamma^{\rho}_{\beta\lambda}$  and treating this equation as a relation between distributions in the manifold, we will find a relation between the Riemann tensor of  $V_4$  and the Riemann tensors defined from  $\Gamma^{+\alpha}_{\beta\gamma}$  and  $\Gamma^{-\alpha}_{\beta\gamma}$ . In fact, from formulas (112) and (113) we have

$$\partial_{\mu}\underline{\Gamma}^{\alpha}_{\beta\lambda} = (\underline{1} - \underline{\theta}) \cdot \partial_{\mu}\Gamma^{-\alpha}_{\beta\lambda} + \underline{\theta} \cdot \partial_{\mu}\Gamma^{+\alpha}_{\beta\lambda} + \delta \cdot n_{\mu} \left[ \Gamma^{\alpha}_{\beta\gamma} \right]$$

and, given that  $\underline{\Gamma}^{\alpha}_{\beta\gamma}$  are distributions associated to functions so that the product  $\underline{\Gamma}^{\alpha}_{\lambda\rho}\underline{\Gamma}^{\rho}_{\beta\mu}$  is well defined

$$\underline{\Gamma}^{\alpha}_{\lambda\rho}\underline{\Gamma}^{\rho}_{\beta\mu} = (\underline{1} - \underline{\theta}) \cdot \Gamma^{-\alpha}_{\lambda\rho}\Gamma^{-\rho}_{\beta\mu} + \underline{\theta} \cdot \Gamma^{+\alpha}_{\lambda\rho}\Gamma^{+\rho}_{\beta\mu}$$

where we have made use of  $\underline{\theta} \cdot \underline{\theta} = \underline{\theta}$  and its consequences  $\underline{\theta} \cdot (\underline{1} - \underline{\theta}) = 0$ ,  $(\underline{1} - \underline{\theta}) \cdot (\underline{1} - \underline{\theta}) = (\underline{1} - \underline{\theta})$ , we finally arrive at the following expression for the Riemann tensor distribution

$$\underline{R}^{\alpha}_{\beta\lambda\mu} = (\underline{1} - \underline{\theta}) \cdot R^{-\alpha}_{\beta\lambda\mu} + \underline{\theta} \cdot R^{+\alpha}_{\beta\lambda\mu} + \delta \cdot n_{\lambda} \left[ \Gamma^{\alpha}_{\beta\mu} \right] - \delta \cdot n_{\mu} \left[ \Gamma^{\alpha}_{\beta\lambda} \right] . \quad (115)$$

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