

Generalized gravitational entropy

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We consider classical Euclidean gravity solutions with a boundary. The boundary contains a non-contractible circle. These solutions can be interpreted as computing the trace of a density matrix in the full quantum gravity theory, in the classical approximation. When the circle is contractible in the bulk, we argue that the entropy of this density matrix is given by the area of a minimal surface. This is a generalization of the usual black hole entropy formula to euclidean solutions without a Killing vector.

A particular example of this set up appears in the computation of the entanglement entropy of a subregion of a field theory with a gravity dual. In this context, the minimal area prescription was proposed by Ryu and Takayanagi. Our arguments explain their conjecture.

1. Introduction

Originally the concept of entropy arose from equilibrium thermodynamics. However, we now think of entropy as a measure of information. In particular, we can assign an entropy to a general density matrix via

$$S = -\text{Tr}[\rho \log \rho] \quad (1.1)$$

By thinking about the thermodynamics of black holes the area formula for gravitational entropy was discovered [1,2,3]. Gibbons and Hawking introduced a thermodynamic interpretation of euclidean gravity solutions with a $U(1)$ isometry [4]. The idea is that one considers Euclidean solutions with prescribed boundary conditions. The boundary conditions, as well as the solutions, are invariant under a $U(1)$ symmetry¹. These solutions can be viewed as describing the computation of the partition function of a quantum theory in the classical approximation. In other words, one thinks of the Euclidean gravitational action as $\log Z(\beta) = -S_{E,grav}$. Then the entropy, obtained as $S = -(\beta\partial_\beta - 1) \log Z$, is equal to the area of the codimension two surface which is a fixed point for the $U(1)$ symmetry in the bulk. Classically, the boundary can be chosen to be any surface where we put boundary conditions. It can also be an asymptotic boundary such as the AdS boundary.

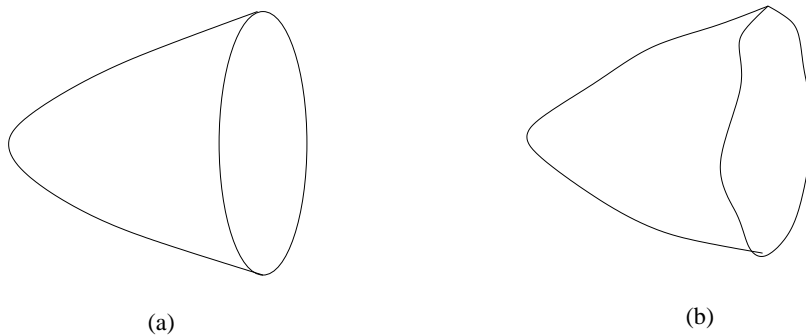


Fig. 1: (a) A euclidean solution with a $U(1)$ symmetry is interpreted as computing the equilibrium thermodynamic partition function of the gravity theory. (b) We consider a euclidean solution with a circle but without a $U(1)$ symmetry. This is interpreted as computing $\text{Tr}[\rho]$ for an un-normalized density matrix in the gravity theory. This is the density matrix produced by euclidean evolution.

¹ Here we assume that there is a single $U(1)$ symmetry, otherwise we need to add the corresponding chemical potentials, etc.

Interestingly, one can extend the notion of gravitational entropy to situations without a $U(1)$ symmetry as follows.

Let us first consider a general quantum system. Its Euclidean evolution generates an un-normalized density matrix

$$\rho = P e^{-\int_{\tau_0}^{\tau_f} d\tau H(\tau)} \quad (1.2)$$

where we considered a general time dependent Euclidean Hamiltonian. We can compute the entropy of this density matrix by the “replica trick”. Namely, first notice that $Tr[\rho]$ can be computed by considering euclidean evolution on a circle, identifying $\tau_f = \tau_0 + 2\pi$ ². Similarly, we can compute $Tr[\rho^n]$ by considering time evolution over a circle of n times the length of the original one, where the couplings in the theory are strictly periodic under shifts of the original circle, $H(\tau + 2\pi) = H(\tau)$.

We then can compute the entropy as

$$S = -n \partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = -Tr[\hat{\rho} \log \hat{\rho}] , \quad (1.3)$$

$$Z(n) \equiv Tr[\rho^n] , \quad \hat{\rho} \equiv \frac{\rho}{Tr[\rho]}$$

where now $\hat{\rho}$ is a properly normalized density matrix. This involves computing $Z(n)$ and then performing an analytic continuation in n .

Going back to the gravitational context, we can consider metrics which end on a boundary. We assume that the boundary has a direction with the topology of the circle. The boundary data can depend on the position along this circle but it respects the periodicity of the circle. We define the coordinate $\tau \sim \tau + 2\pi$ on the circle. We can then consider a spacetime in the interior which is smooth. Its Euclidean action is defined to be $\log Z(1)$. See fig. 1(b). We can also consider other spacetimes where we take the same boundary data but consider a new circle with period $\tau \sim \tau + 2\pi n$. Their Euclidean action is defined to be $\log Z(n)$. These computations can be viewed as computing $Tr[\rho^n]$ for the density matrix produced by the Euclidean evolution. See fig. 2. If we are sufficiently diligent, we can find these actions, analytically continue in n the corresponding answers and compute S as in (1.3). This has been explicitly done in [5,6] for some examples in three dimensional gravity.

Note that we are implicitly assuming that gravity is holographic. We are imagining that setting boundary conditions on some boundary defines the theory and that the interior

² Throughout this paper we set the coordinate length of the initial circle to 2π . Of course, its physical length depends on the metric.

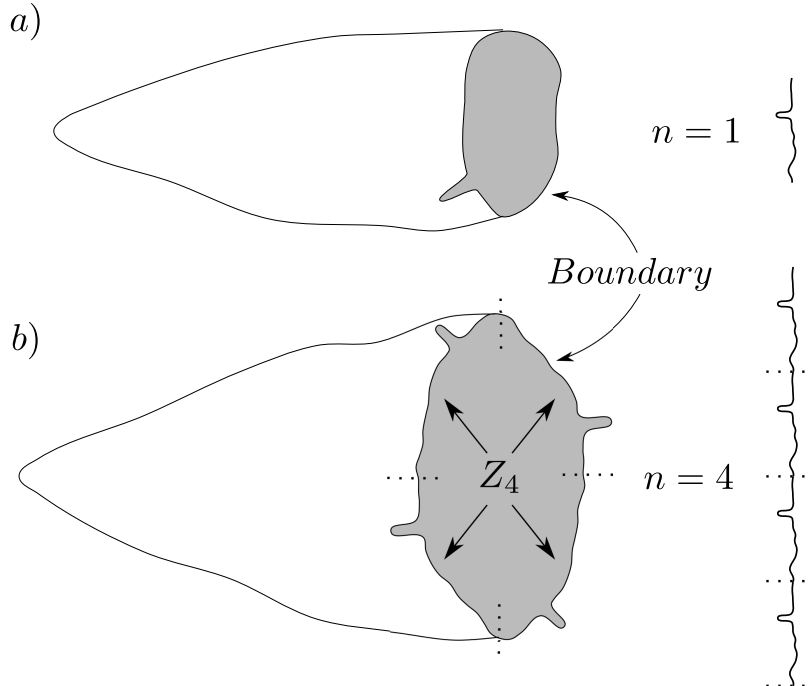


Fig. 2: Computing the entropy using the replica trick. (a) Euclidean solution for $n = 1$. (b) Solution for $n = 4$. At the boundary we go around the original circle n times before making the identification. We then find a smooth gravity solution with these boundary conditions. The curves in the right hand side are schematically giving the boundary conditions at infinity. We see that in (b), we simply repeat n times the boundary conditions we had in (a).

geometry is an approximation to the full computation. We do not know how (and whether) holography works for general boundaries. Here we only need it to be approximately valid so that this classical computation has the interpretation of computing an approximate density matrix in some approximate theory. In cases where the boundary is a true asymptotic boundary (such as a locally asymptotically AdS boundary) the situation is well understood. This corresponds to computing the entropy of a perfectly well defined density matrix in the dual field theory .

Interestingly, there is a simple conjecture for the final answer. The entropy is also given by the area of a special codimension two surface in the bulk of the original ($n = 1$) solution. At this surface the circle shrinks smoothly to zero size. The surface obeys a minimal area condition.

$$S \equiv - n \partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = \frac{A_{\text{minimal}}}{4G_N} \quad (1.4)$$

From now on, $\log Z(n)$ denotes the classical gravity action $\log Z(n) = -S_{\text{Grav}}$ of the n^{th} solution. This formula was first conjectured by Ryu and Takayanagi in the context of the computation of entanglement entropy of conformal field theories with gravity duals [7](see [8] for a review³). Proving their formula amounts to proving the above conjecture, as we explain below. Notice that (1.4) can be viewed as a statement about classical general relativity. It is a relation between the actions for classical solutions that are produced by the replica trick and the area of the minimal area solution with $n = 1$. Of course, for solutions with a $U(1)$ symmetry, (1.4) reduces to the standard Gibbons-Hawking computation. In that case, the $U(1)$ symmetry also ensures that the horizon is a minimal surface, with zero extrinsic curvature.

In this paper we will give an argument for (1.4) based on reasonable assumptions regarding the analytic continuation of the solutions away from integer values of n .

We will also explain why proving (1.4) is equivalent to proving the Ryu Takayanagi conjecture. The Ryu-Takayanagi conjecture for the case of asymptotically AdS_3 pure gravity was proven in [5,6]. Previous arguments include [11], whose assumptions were criticized in [12].⁴

This paper is organized as follows. In section 2 we perform some explicit computations in a simple example. In section 3 we review the derivation of the entropy formula for the case with a $U(1)$ symmetry. In section 4 we present the arguments for the main formula (1.4). There we explain how the solution looks for n close to one. We also derive the minimal area condition for the surface. In section 5 we discuss the connection to entanglement entropy in field theories with gravity dual. In section 6 we present the conclusions. In the appendices we present some further explicit examples and more details on the computations.

2. A simple example without a $U(1)$ symmetry

Since our discussion has been a bit abstract, let us discuss a very simple concrete example. This example will also motivate some assumptions that we will make later.

³ See [9,10] for related work.

⁴ Fursaev [11] took the solution for $n = 1$ and set $\tau \sim \tau + 2\pi n$ everywhere in the bulk. This introduces a conical singularity in the bulk. As noted by Headrick [12], for integer n , one should instead consider solutions which are non-singular in the bulk.

Let us start with the BTZ geometry

$$ds^2 = \left[\frac{dr^2}{(1+r^2)} + r^2 d\tau^2 + (1+r^2) dx^2 \right] \quad (2.1)$$

This metric has a $U(1)$ isometry along the circle labeled by τ , $\tau \sim \tau + 2\pi$. All functions will be invariant under translations in x . This direction will not play any role in this discussion and we take it to be compact of size L_x . Computing the entropy for this solution gives the standard area formula, S_0 , for this solution.

We now add a complex, minimally coupled, massless scalar field ϕ . We set boundary conditions that are not $U(1)$ invariant

$$\phi = \eta e^{i\tau} , \quad \text{at } r = \infty \quad (2.2)$$

We now compute the gravitational action to second order in η for the family of solutions described above. The metric is changed at order η^2 , but since the original background obeys Einstein's equations, there is no contribution from the gravitational term to order η^2 . So, to this order, the whole contribution comes from the scalar field term in the action.

Namely, for the n^{th} case, we need to consider a spacetime with the same boundary conditions as in (2.2) but where $\tau \sim \tau + 2\pi n$. This implies that the spacetime in the interior is

$$ds^2 = \left[\frac{dr^2}{(n^{-2} + r^2)} + r^2 d\tau^2 + (n^{-2} + r^2) dx^2 \right] \quad (2.3)$$

And we need to consider a scalar field in this spacetime. We can write the wave equation. The solution of the wave equation that is regular at the origin and obeys (2.2) at infinity is

$$\phi = \eta e^{i\tau} f_n(r) , \quad f_n(r) = (nr)^n \frac{\Gamma(\frac{n}{2} + 1)^2}{\Gamma(n+1)} {}_2F_1 \left(\frac{n}{2}, \frac{n}{2} + 1; n+1; -(nr)^2 \right) \quad (2.4)$$

Note that $f_n \rightarrow 1$ as $r \rightarrow \infty$.

We now evaluate the gravitational action for every n . We evaluate it to second order in η , so we consider the quadratic action for the field ϕ . Using standard formulas we can write

$$\begin{aligned} \log Z(n)|_{\eta^2} &= - \int_{AdS_3} |\nabla\phi|^2 = -(2\pi n)L_x [r^3 \phi^* \partial_r \phi]_{r=\infty} = \\ &= (2\pi L_x) [1 - n \log n + n\psi(n/2) + (\text{linear in } n)] \end{aligned} \quad (2.5)$$

where L_x is the length of the x direction and ψ is the Euler ψ function. The terms linear in n include divergent terms that should be subtracted. However, they do not contribute to the entropy (1.3).

We analytically continue in n and compute the entropy via (1.3) to find

$$S = S_0 + \eta^2 \pi L_x \left(4 - \frac{\pi^2}{2}\right) \quad (2.6)$$

We can now compare this with the answer we expect from the area formula. This non-zero configuration for the scalar field changes the geometry to second order in η . Thus it produces a second order change in the area of the horizon. This change can be computed from Einstein's equation. We obtain the same answer (2.6). This is done in detail in appendix A, where we also consider a scalar field with an arbitrary mass.

So, we have explicitly checked the conjecture for this special case. Now, let us make some remarks.

We considered a complex scalar field, but the computation can be done also for a real scalar field with boundary conditions $\phi = \eta \cos \tau$ at infinity. The result is essentially the same. See appendix A.

Notice that the solution for the n^{th} case has a Z_n symmetry. This is a replica symmetry of the boundary conditions which extends to the bulk solution. So, in this case we are not breaking the replica symmetry. Notice that $r = 0$ is a fixed point of the action of the Z_n replica symmetry for all $n > 1$. In this case, the metric has a $U(1)$ symmetry. However, the full scalar field configuration is only symmetric under the Z_n ⁵.

Here we have computed $\log Z(n)$ and then analytically continued the answer. The geometry (2.3) is well defined also for non-integer n and we can trivially continue it to non-integer values of n , and it remains smooth. We could ask whether we can also analytically continue the whole field configuration to non integer values of n . Notice that as we vary n , the τ dependence at the boundary is kept fixed. Thus, even for non-integer n , we will keep the same boundary condition. This boundary condition is not compatible with a non-integer period for τ . We will ignore this. In other words, we will integrate τ between $[0, 2\pi]$ and multiply the result by n . However, as we go to $r = 0$, we find that the scalar field behaves as $\phi \sim r^n e^{i\tau}$, which leads to a singularity for the scalar field at $r = 0$. The

⁵ In this case there is a $U(1)$ symmetry which is shift in τ combined with a phase rotation of the complex field. But for a similar computation with a real scalar field we only have the Z_n symmetry.

scalar field, or its stress tensor do not diverge if $n > 1$. In other words, this appears to be a relatively harmless integrable singularity. This singularity seems physically questionable. But we are not trying to give a physical interpretation to the solution with non-integer n . We are only trying to define it mathematically, as an intermediate step in computing the replica trick answer. One could worry that if we allow singularities, then the solution will not be uniquely defined. However, we are allowing a very specific behavior which determines a unique solution for given boundary conditions. More explicitly, note that when we solve the wave equation near $r \sim 0$, we get two solutions $r^n e^{i\tau}$ and $r^{-n} e^{i\tau}$. We set to zero the coefficient of the second solution at the origin. This prescription uniquely selects a solution, both for integer and non-integer n .

This gravity theory in AdS_3 with a massless scalar field can arise from a Kaluza Klein reduction of a higher dimensional theory. For example, it can come from a ten dimensional solution of the form $AdS_3 \times S^3 \times T^4$. Then the massless field can be an off-diagonal component of the metric on of the four torus [13,14]. More explicitly, we can deform the metric of the four torus as

$$ds_{T^4}^2 = e^{2\phi_1} dy_1^2 + e^{2\phi_2} dy_2^2 + e^{-2\phi_1} dy_3^2 + e^{-2\phi_2} dy_4^2 \quad (2.7)$$

where $\phi = \phi_1 + i\phi_2$. We see that the singularity of the field ϕ at the origin translates into a singularity for some of the Riemann tensor components. Let us consider $n = 1 + \epsilon$. Then since $\phi \propto r^{1+\epsilon} e^{i\tau}$ this leads to a singularity in some of the Riemann tensor components $R_{\alpha i \beta i} \sim \frac{\epsilon}{r}$ (no sum over i), where i are the directions on the four torus and α denotes the directions along the two transverse components (labeled also by r and τ) Despite these singularities the action is finite, as we saw when we computed it explicitly.

An alternative way to view the solution labeled by n is the following. We consider the τ circle to have period 2π but introduce a conical singularity at the origin with opening angle $2\pi/n$. This is not the same as the gravity solution with $n = 1$ since the field configuration has to adjust to the presence of the conical singularity. Then, when we evaluate the gravitational action, we integrate τ over $[0, 2\pi]$ but multiply the resulting answer by a factor of n . This factor of n arises because the real period of τ is $2\pi n$ instead of 2π . It is important that we evaluate the gravitational action *without introducing any contributions from the tip of the conical singularity*, since the full space (with the right period for τ) is non-singular^{6 7}. This picture makes sense both for integer or non-integer n .

⁶ This looks superficially similar to what was discussed in [11], but it is different in detail.

⁷ This point of view was also suggested to us by T. Faulkner.

3. Computation of gravitational entropy when there is a $U(1)$ symmetry

In this section we will describe the computation of the entropy using Euclidean methods in a way that it emphasizes the fact that the contribution comes from the horizon. This has been discussed by various authors in a similar form [2,15,16,17,18,19]. Here we say it two ways that we particularly liked.

3.1. Entropy from rounded off cones

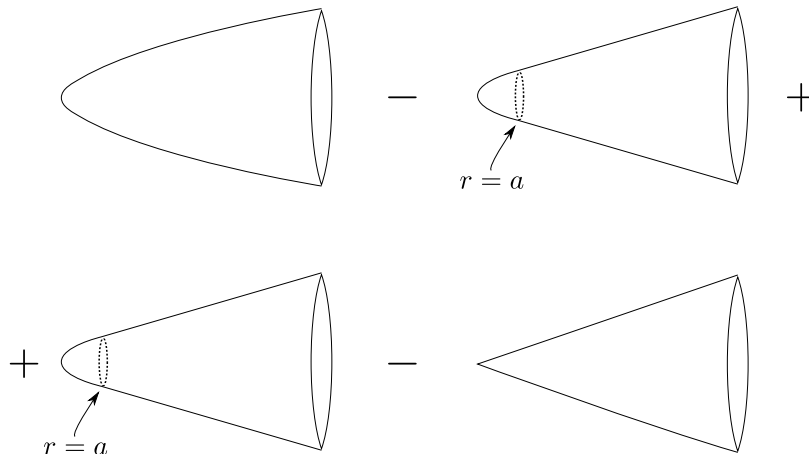


Fig. 3: A particular combination of geometries that is useful for computing the entropy. The first geometry is the correct solution with period $2\pi n$. The last geometry contains a conical singularity. It is the solution with $n = 1$ but with the circle identified after $\tau \rightarrow \tau + 2\pi n$. For $n = 1$ the deficit angle of the cone is very small and it has been greatly exaggerated here for artistic reasons. The two middle ones are identical and correspond to a regularized version of the last solution. They only differ for $r < a$, where a is small regulator. This is not a solution, it is an off-shell configuration. All of the configurations obey the same boundary conditions at infinity.

Setting the period of the circle to be $\tau \sim \tau + 2\pi n$, then we find that the formula for the entropy can be written as

$$S = -n\partial_n [\log Z(n) - n \log Z(1)]_{n=1} \quad (3.1)$$

Let us consider this expression for n close to one. We interpret the first term in the square brackets as the correct, smooth solution when n is not one. We interpret the second term as the solution for $n = 1$ but with a τ which has period $\tau \sim \tau + 2\pi n$. This solution has a conical singularity at the origin. However, we *do not* include any contribution from the

conical singularity. We simply integrate the gravitational action density away from the tip.

We now evaluate the difference in the square brackets in (3.1) by adding and subtracting a smooth geometry which is the same as that of the cone far away from the origin, but it is a regularized cone near the origin, see fig. 3. This smooth geometry is not a solution of the equations of motion of the theory, it is an off-shell configuration. We are simply introducing it to help us perform the computation. It is possible to choose this off-shell configuration in such a way that the metric differs only by an amount of order $n - 1$ from the true solution.

Thus we get

$$S = -n\partial_n \left[(\log Z(n) - \log Z^{\text{off}}(n)) + (\log Z^{\text{off}}(n) - n \log Z(1)) \right]_{n=1} \quad (3.2)$$

Each of the terms in the brackets is the action for one of the configurations in fig. 3. Since the off shell configuration that corresponds to a regularized cone differs by a first order term in $n - 1$ from a solution of the equations, we see that we can interpret the first parenthesis as the result of doing a first order variation away from a solution (the solution with period n). This first order variation vanishes due to the equations of motion for the solution with period n . Notice that both metrics obey the same boundary conditions at the boundary, so that there are no boundary terms⁸.

So all that remains is the second parenthesis. The second parenthesis contains the difference between a smooth cone and a regularized cone. This receives a contribution only from the region near the tip of the cone. This contribution is extensive in the area of the horizon, namely the area of the surface transverse to the tip of the cone. The region near the rounded tip of the cone contains an integral of $\int d^2x \sqrt{g} R$ along the cone directions which gives

$$\int_{\text{Reg Cone}} d^2x \sqrt{g} R \sim 4\pi(1 - n) \quad (3.3)$$

Thus, the final answer has the form

$$S = \frac{1}{16\pi G_N} (\text{Area}) \left(-n\partial_n \int_{\text{Reg Cone}} d^2x \sqrt{g} R \right) = \frac{\text{Area}}{4G_N} \quad (3.4)$$

⁸ The absence of boundary terms is clearest if we write the action in a non-manifestly covariant form using only first derivatives of the metric. Then the fact that the two configurations obey the same boundary conditions for the metric implies that there are no boundary terms.

One can consider a metric that explicitly regularizes the cone, such as [18]

$$ds^2 = dr^2 g^2(r) + r^2 d\tau^2 \quad (3.5)$$

where $g = n + o(r^2)$ at $r \sim 0$ and $g = 1$ for $r > a$, where a is a small distance which sets the size of the regularization. Inserting this metric into the gravitational action we get (3.3). One can choose a completely explicit function such as $g = 1 + (n - 1)e^{-r^2/a^2}$, for example. In this case we can see explicitly that the metric perturbation is of order $(n - 1)$.

3.2. Entropy from apparent conical singularities

Another way to think about this problem is as follows. First we note that, since the solutions are invariant under time translation, the evaluation of $\log Z(n)$ is the same as

$$\log Z(n) = n[\log Z(n)]_{2\pi} \quad (3.6)$$

where $[\log Z(n)]_{2\pi}$ is the gravitational action density for the solution labeled by n but integrated over τ from $[0, 2\pi]$ (instead of $[0, 2\pi n]$). We can now write the entropy as

$$S = -n^2 \partial_n [\log Z(n)]_{2\pi} \quad (3.7)$$

Note that the solution labeled by n is a smooth geometry if the τ circle has period $2\pi n$. On the other hand, imagine we wanted to view it as a configuration where the τ period continues to be 2π . In that case, it is a geometry with a conical singularity whose opening angle is $2\pi/n$. Thus we can view

$$[\log Z(n)]_{2\pi} \quad (3.8)$$

as the gravitational action of a configuration with $\tau = \tau + 2\pi$ but with a conical singularity with opening angle $2\pi/n$, *without including any curvature contribution from the conical singularity*. Then we see that the expression of the entropy (3.7) involves taking a derivative with respect to n . When we change n we are changing the opening angle of the singularity. In addition, we are changing the metric and other fields everywhere since they have to adjust to this new strength of the singularity. However, since the original solution (the solution with $n = 1$) is a solution of the equations, we would naively expect that a first order variation of the metric and other fields should vanish due to their equations of motion. This naive expectation is essentially right, except for the fact that we are changing the boundary conditions at the origin, since the strength of the conical singularity is being

changed. Thus, the only change in the action comes from a boundary term. In other words, when we change n the action changes as

$$\begin{aligned}
-\partial_n[\log Z(n)]_{2\pi|_{n=1}} &= \int E_g \partial_n g + E_\phi \partial_n \phi + \\
&+ \frac{1}{8G_N} \int_{r \sim 0} dy^{D-2} \sqrt{g} (\nabla^\mu \partial_n g_{\mu r} - g^{\mu\nu} \nabla_r \partial_n g_{\mu\nu}) = \frac{A}{4G_N}
\end{aligned} \tag{3.9}$$

where E_g and E_ϕ are the equations of motion for the metric and other fields, which vanish. Here y are the coordinates along the $r = 0$ surface. The boundary term vanishes at the large r boundary since we are choosing boundary conditions in such a way that the variation of the action gives the equations of motion without extra boundary terms. On the other hand, at the horizon (at $r = 0$), we do get a contribution from the boundary term. This boundary term produces the area contribution. Note that the n derivatives of the metric are evaluated at the horizon. For example, in the parametrization $ds^2 = n^2 dr^2 + r^2 d\tau^2$ near the origin, we get, as the only non-vanishing component, $\partial_n g_{rr}|_{n=1} = 2$. With these expressions we can evaluate the parenthesis in (3.9) and obtain $2/r$.

This derivation easily generalizes to theories with higher derivative actions, giving the Wald entropy [20,21,22].

Note that in both cases we used explicitly the locality of the action along the τ direction. It would be interesting to find the corresponding formula in weakly coupled string theory exactly in α' .

4. Argument for the entropy formula (1.4)

4.1. Properties of the metric for n integer

For $n = 1$, the boundary contains a circle which we label by the coordinate τ . Recall that the boundary is the surface where we are putting boundary conditions. This circle is non-contractible on the boundary, but it can be contractible in the interior of the geometry. Here by boundary, we mean the boundary where we set boundary conditions for the gravitational action. It need not be an asymptotic boundary.

The metric and all fields are periodic on this circle. Let us collectively denote these fields as $\psi(\tau)$, with

$$\psi(\tau) \sim \psi(\tau + 2\pi) \tag{4.1}$$

Of course, the fields depend on other coordinates, but here we are highlighting their τ dependence. We impose boundary conditions

$$\psi(\tau)|_{\text{Boundary}} = \hat{\psi}_B(\tau), \quad \hat{\psi}_B(\tau) = \hat{\psi}_B(\tau + 2\pi) \quad (4.2)$$

where we specify the functions $\hat{\psi}_B(\tau)$, which are periodic.

The solution with $n > 1$, has exactly the same boundary conditions (4.2), but we require the periodicity $\tau = \tau + 2\pi n$ on the τ circle. This implies that the boundary conditions have a Z_n symmetry. We assume that the bulk solution continues to have this Z_n symmetry.⁹

Each of the solutions for $n > 1$ has a special codimension two surface which is left invariant by the action of Z_n . We will focus on this surface. We can choose a coordinate r which is a radial coordinate away from this surface and an angle τ . The true angle around the surface is really $\alpha = \tau/n$, we have chosen τ to have the same period as the one we have at infinity ($\alpha \sim \alpha + 2\pi$). The metric in the two directions transverse to this surface has the form

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 + \dots \quad (4.3)$$

where the factor of n comes from demanding that there is no singularity at $r = 0$. In addition, all fields are required to have an $e^{ik\tau}$ dependence, with integer k . This comes from the period of τ and the Z_n symmetry. Thus, a scalar field would behave as $r^n e^{i\tau} \sim r^n e^{in\alpha}$ near the origin, as results from demanding that it is non-singular.

⁹ In principle, the replica Z_n symmetry can be broken. Our discussion assumes that it is not broken. The simplest gravity solutions can also develop other instabilities. For example, if one considers gravity in AdS_{d+1} with a boundary $H_{d-1} \times S^1$. If the radius of S^1 is equal to the radius of H_{d-1} then the full solution is AdS , viewed as a black brane with a hyperbolic spatial section. If we make the S^1 n times larger, then for large n , we approach an extremal black hole with an $AdS_2 \times H_{d-1}$ near horizon geometry. This can lead to bad tachyons for $m^2 R_{AdS_{d+1}}^2 < -d/4$. Thus if the original AdS_{d+1} has tachyons in the allowed range $-d^2/4 \leq m^2 R_{AdS}^2 < -d/4$, then we will have an instability. This is similar to the discussion of [23], where an extremal Reissner-Nordstrom black brane in AdS_4 , with a near horizon geometry $AdS_2 \times R^2$ was considered. Good tachyons in AdS_4 can be bad tachyons in the AdS_2 region if $-\frac{9}{4} < m^2 R_{AdS}^2 < -\frac{3}{2}$. See [24] for further discussion. Here we assume that we have no dangerous tachyons that can lead to these instabilities. Similar instabilities were observed computing the Renyi entropies of circular regions in the three dimensional interacting $O(N)$ model [25].

As a side remark, notice that if the bulk space has no fixed points under the Z_n action, then this means that we can choose the coordinate τ in the interior so that this circle never shrinks. An example is a space with topology $R^{d-1} \times S^1$, but with a metric that depends on the coordinate along the S^1 . In these cases the entropy is zero. The reason is very simple, the solution for the n^{th} replica is the same as the solution with $n = 1$ but with a longer circle so that $\log Z(n) = n \log Z(1)$. Here, of course, we used the locality of the classical action.

4.2. Metric for n non-integer

Here we make some assumptions on the form of the metric when n is not an integer. We will continue to impose exactly the same boundary condition (4.2), which is periodic with period 2π . This is *not* compatible with $\tau \rightarrow \tau + 2\pi n$. Now, in the region where the τ circle has positive size, we can ignore this problem and think of τ as being non-compact. When we evaluate the action, we can integrate the τ direction from 0 to 2π and then multiply by n .

However, we expect that there is still a surface where the τ circle shrinks to zero. For the two dimensions transverse to this surface we impose that the metric continues to behave as in (4.3), even though n is not an integer. The rest of the fields, including other components of the metric, are chosen so that they are periodic in $\tau \rightarrow \tau + 2\pi$ as in (4.1). This implies that the field configuration is singular at $r = 0$. However, we expect that this singularity is as harmless as the one we had for the scalar field in section 2 .

This seems a reasonable assumption. As evidence for its validity we can point to the explicit example mentioned in section 2 .

An equivalent way to specify the solutions is to compactify the τ circle to $\tau + 2\pi$ in all cases (all values of n) and demand that there is a conical defect angle with opening angle $2\pi/n$ in the interior. We do not introduce any contribution to the action from the tip of the cone. In addition, we multiply the gravitational action by a factor of n . This is mathematically equivalent to what was discussed above and the reader can choose the preferred interpretation.

Note that this is similar to introducing a cosmic string (or cosmic $D - 3$ brane) with opening angle $2\pi/n$ in the original solution, with the metric backreacting as necessary to account for its presence.

As $n \rightarrow 1$ the solution goes over to the solution with $n = 1$. Thus, this analytically continued solution is close to the $n = 1$ solution and we can expand it in powers of $n - 1$.

4.3. Derivation of the minimal surface condition

We emphasized that for $n > 1$ we have a special surface where the circle shrinks, and is fixed under the Z_n action. But for $n = 1$ there is no obvious special surface, since there is no unique way to choose the coordinate τ in the interior once it is not associated to a $U(1)$ isometry. So, when we expand the solution in $n - 1$ we need to select a surface. Motivated by the Ryu-Takayanagi conjecture we want to select a minimal area surface. In this subsection we will explain the origin of this minimal area condition. The final conclusion is that the condition comes from demanding that the solution obeys the Einstein equations to leading order in $n - 1$. This derivation is essentially the same as the derivation of the equations of motion for a cosmic string (or $D - 3$ brane) from the behavior of the metric near the conical singularity. This problem was analyzed previously in [26,27].

Two dimensional dilaton gravity

It is good to start with a simple situation first. For that purpose we will consider a two dimensional dilaton gravity where the action is

$$-S_{\text{Grav}} = \frac{1}{16\pi} \int d^2x \sqrt{g} e^{-2\varphi} [R + 4(\nabla\varphi)^2 + \dots] \quad (4.4)$$

where the dots indicate other fields, or a potential for φ , etc. Notice that if we have a solution with a horizon, then $e^{-2\varphi}$ at the horizon plays the role of the area in Planck units of the higher dimensional gravity solutions. In this case the codimension two surface is just a point. The minimum area condition is that $e^{-2\varphi}$ is a minimum (or really an extremum) at this point. We will derive this condition from demanding that the configuration for small $\epsilon \equiv n - 1$ obeys the linearized field equations near $r = 0$. In other words, expanding the fields around the $n = 1$ solution, and *assuming the periodicity condition for the fields*, (4.1), we will see that we can only obey the equations if $\partial_i\varphi = 0$.

Let us say that as $n \rightarrow 1$ the special surface goes over to some point of the $n = 1$ manifold. Let us pick this point to be the origin in some coordinate system x^1, x^2 . Then the metric of the $n = 1$ solution around this point is $ds^2 = dx_1^2 + dx_2^2 + o(x^2)$. The field φ is regular at this point. Now, for $n - 1 = \epsilon$ we expect a metric of the form $ds^2 = e^{2\rho}(dx_1^2 + dx_2^2)$, with $e^{2\rho} = r^{2(\frac{1}{n}-1)}$, as $r \rightarrow 0$. Then to first order in ϵ we have $\delta\rho \sim -\epsilon \log r$ to be the first order solution. We consider the two following equations for two dimensional dilaton gravity

$$\begin{aligned} 0 &= e^{-2\varphi} (4\partial_z\varphi\partial_z\rho + 2\partial_z^2\varphi) + T_{zz}^{\text{matter}} \\ 0 &= e^{-2\varphi} (4\partial_{\bar{z}}\varphi\partial_{\bar{z}}\rho + 2\partial_{\bar{z}}^2\varphi) + T_{\bar{z}\bar{z}}^{\text{matter}} \end{aligned} \quad (4.5)$$

where $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. Here T^{matter} denotes the stress tensor for the rest of the fields of the theory, coming from the dots in (4.4). Expanding the first equation to first order we find

$$-2\partial_z\varphi(0)\frac{\epsilon}{z} + 2\partial_z^2\delta\varphi + \delta T_{zz}^{\text{matter}} = 0 \quad (4.6)$$

and a similar equation by expanding the second. Here $\partial_z\varphi(0)$ is the derivative of the field for the $n = 1$ solution at the origin. It is just a z independent constant. Since the matter stress tensor is not expected to be singular at order $1/r$, we find that

$$\partial_z^2\delta\varphi \propto \frac{\epsilon\partial_z\varphi(0)}{z}, \quad \partial_{\bar{z}}^2\delta\varphi \propto \frac{\epsilon\partial_{\bar{z}}\varphi(0)}{\bar{z}} \quad (4.7)$$

up to terms that are less singular as $r \rightarrow 0$. Now we assume that the solution for $\delta\varphi$ has a fourier expansion with integer powers of $e^{i\tau}$. The first equation in (4.7) suggests that we try a solution proportional to $\delta\varphi \propto z \log z$. However, the periodicity condition under shifts of τ suggests that we should consider $\delta\varphi \propto z \log(z\bar{z})$. However, this is not a solution of the second equation. Thus this implies that the gradients of the field should vanish at the origin.

More formally, we can argue as follows. The periodicity condition implies that if we take the τ derivative of any field and integrate over τ between zero and 2π , we should get zero. This is true both for $\delta\varphi$ and its derivatives. In particular, note that the following combination of derivatives gives

$$\partial_\tau [(r\partial_r - 1)\partial_z\delta\varphi] \propto (z\partial_z - \bar{z}\partial_{\bar{z}})(z\partial_z + \bar{z}\partial_{\bar{z}} - 1)\partial_z\delta\varphi \propto \epsilon\partial_z\varphi(0) \quad (4.8)$$

where we used both equations in (4.7). Now the integral over τ of (4.8) should be zero, according to our assumption about the periodicity of $\delta\varphi$. This then implies that $\partial_z\varphi(0) = 0$.

In summary, in this case we found that the condition comes from the zz and $\bar{z}\bar{z}$ components of the Einstein equations. In higher dimensions we expect that this will come from Einstein's equations in the directions normal to the surface.

Note that if we changed the coefficient of the dilaton kinetic term in (4.4) from $4(\nabla\varphi)^2$ to $(4 + \sigma)(\nabla\varphi)^2$, then we would be adding terms of the form $\sigma\partial_z\varphi\partial_z\varphi$ to the equations in (4.5). Expanding around the background solution such terms lead to contributions that are subleading, in the expansion around the origin, compared to the terms already taken

into account in (4.7). Thus, if we had a two dimensional action with a different coefficient for the dilaton kinetic term, we would have reached the same conclusion¹⁰.

Einstein gravity in D dimensions

We now go back to the case of Einstein gravity. In general, we can expand the metric of the $n = 1$ solution around the special surface as

$$\begin{aligned} ds^2 &= dr^2 + R^2 d\tau^2 + b_i d\tau dy^i + g_{ij} dy^i dy^j , \\ g_{ij} &= h_{ij} + r \cos \tau K_{ij}^1 + r \sin \tau K_{ij}^2 + o(r^2) \\ R &= r + o(r^3) , \quad b_i = o(r^2) \end{aligned} \tag{4.9}$$

where r is coordinate normal to the surface and y_i are coordinates along the surface. Here K_{ij}^α are the two extrinsic curvature tensors. h_{ij} depends only on y_i but not on r or τ . When we deform away from $n = 1$ we assume that we cannot change the period of the cosines above.

When $n = 1 + \epsilon$ some of the metric components generically go like $r^{1+\epsilon}$. This can give rise to terms in the equations of motion going like $1/r$. These terms can only come from situations where we have two derivatives along the transverse directions (the r and τ directions). Such terms in the equations of motion are the same as the ones we would obtain by performing a dimensional reduction from D dimensions to the two transverse directions. This brings us back to the previous case. More explicitly, we write the full D dimensional metric as

$$ds^2 = e^{2\rho}(dx_1^2 + dx_2^2) + e^{-\frac{4\varphi}{D-2}} \hat{g}_{ij} dy^i dy^j + o(r^2) , \quad \det(\hat{g}_{ij}) = 1 \tag{4.10}$$

where \hat{g}_{ij} is the transverse metric appearing in (4.9) but rescaled so that its determinant is one. The off diagonal terms in (4.9) do not contribute to terms of order $1/r$ in the equations of motion. Both \hat{g}_{ij} and φ depend on all the coordinates, the y_i and the x_i . Here we have pulled out the overall volume factor of the transverse space and parametrized it by φ . Dimensionally reducing to the first two dimensions gives us (4.4), but with a different coefficient for the dilaton kinetic term. Thus, we obtain the same conditions that $\partial_{x^\alpha} \varphi = 0$ for $\alpha = 1, 2$. Now if we translate between φ and the original metric (4.9) we find that

$$-4\varphi = \log(\det(h_{ij})) + x^1 K^1 + x^2 K^2 + o(r^2) , \quad K^\alpha = h^{ij} K_{ij}^\alpha \tag{4.11}$$

¹⁰ This is to be expected since this coefficient can be changed by a field redefinition of the metric.

where K^α are the traces of the extrinsic curvature tensors. We then see that the condition $\partial_\alpha \varphi = 0$ implies that

$$K^1 = K^2 = 0 \tag{4.12}$$

Namely, the traces of the extrinsic curvatures should vanish. There are two directions that are transverse to the surface so we have two relevant extrinsic curvatures. These coincide with the equations of motion for a minimal area surface. There are two transverse directions to the surface and thus two equations. In appendix B we derive (4.12) directly in D dimensions, without doing the dimensional reduction.

Note that the non-trace part of the extrinsic curvatures are *not* constrained to vanish. In fact, already in our simple example of section 2 we have non-vanishing extrinsic curvature if we interpret the scalar field as coming from a component of a higher dimensional metric as in (2.7).¹¹

4.4. Computation of the entropy using the cone method

Once we have established the form of the solution, we can compute the entropy using the cone method as explained in section 3. The arguments are similar, but one has to check that the mild singularities we discussed above cause no problems.

Let us discuss this first for the case of *AdS* plus a scalar field discussed in section 2. There, the singularity is only present in the scalar field which behaves as $\phi \sim r^\epsilon r e^{i\tau}$. With this mild singularity, if we integrate by parts in order to use the equation of motion for ϕ , it is clear that we do not run into any problem at $r = 0$. The most dangerous term seems to come from the variation of $\delta \int dr r (\partial_r \phi)^2 = 2 \int dr r \partial_r \phi \partial_r \delta \phi \rightarrow r \partial_r \phi \delta \phi|_0$. However, $\delta \phi$ would also vanish at the origin if we are considering the variations that come from varying n in the solution. In other words, when we compare the correct configuration with $n - 1 > 0$ and a regularized cone, we can consider a regularized cone where ϕ has the same type of singularity at the origin. This shows that the first parenthesis in (3.2) vanishes.

The second parenthesis only gives us something interesting if we consider terms that have two derivatives acting on the metric, otherwise their contribution is going to be small as we remove the regulator. Thus, only the metric in the two directions transverse to the

¹¹ More explicitly, in the notation of that section, if the field at the origin goes as $\phi = r e^{i\tau}$ this leads to $\phi_1 = r \cos \tau = x^1$ and there is an extrinsic curvature component $K_{y^1 y^1}^1 = -K_{y^3 y^3}^1$ using the coordinates (2.7) and (2.1) for the 3-d part.

minimal surface are relevant. And in those dimensions the computation reduces to the usual one, with the contribution coming only from the curvature term in the action.

In conclusion, evaluating the differences in (3.2) we find that the answer is equal to the area, as we wanted to prove to argue (1.4).

The discussion so far was completely local in the directions transverse to the “horizon”. Here by horizon we mean the point in the two transverse directions where the circle is shrinking to zero size. In some cases this “horizon” can have multiple disconnected regions. Then, we should sum over the areas of each of the horizons. Even when we have multiple horizons, the period of the τ circle is the same in the whole solution.

4.5. A comments on other $U(1)$ symmetries

Throughout this discussion we have focused on the particular geometric circle that we used to define the density matrix and the replica trick. We considered cases where we have no translation symmetry along the circle. However, we can have other $U(1)$ symmetries. As a simple example, we can have a $U(1)$ gauge field in the bulk. Then as in the ordinary case, [4], the integral of the gauge field along the τ circle should vanish at the origin $\int_0^{2\pi n} d\tau A_\tau \Big|_{r \rightarrow 0} = 0$ (up to global gauge transformations), where the τ circle shrinks. This should hold for all n , both integer and non-integer. At the boundary we can fix the holonomy of A along the τ circle as we please. In order to compute the entropy of the density matrix with a chemical potential we should fix the integral $\hat{\mu} \equiv \int_0^{2\pi} A_\tau$ at the large r boundary. If we keep everything else fixed at the boundary but we vary $\hat{\mu}$, this has the interpretation of changing the density matrix $\rho \rightarrow e^{i\hat{\mu}Q}\rho$ where Q is the charge associated to the $U(1)$ symmetry. We can compute the entropy of this density matrix by treating this boundary condition as we treated all other boundary conditions. Namely, $A_\tau(\tau)$ is kept fixed. Therefore its circulation over the τ circle of length $2\pi n$ is $n\hat{\mu}$. Of course, if the holonomy in the τ circle is different at the origin ($r = 0$) than at the boundary, then we will have a non-zero field strength in the bulk. The computation of the entropy is identical to what we discussed in general.

This other $U(1)$ symmetry can also be an ordinary geometric isometry, and its treatment is similar.

5. Connection to the Ryu-Takayanagi formula

We presented the computation of the entropy of the gravitational density matrix in a form that is very general. The objective was to emphasize that (1.4) is really a statement about an analytic continuation of classical solutions. In this section we explain why the conjecture (1.4) for the entropy is related to the Ryu-Takayanagi formula for entanglement entropy.

The Ryu-Takayanagi formula is a conjecture in the AdS/CFT context [7]. In the quantum field theory one is interested in computing the entanglement entropy of a spatial region A on the boundary of the field theory. This spatial region has a boundary ∂A . The conjecture is that this entanglement entropy is given by the area (in Planck units) of a codimension two minimal surface in the bulk whose boundary ends on ∂A .

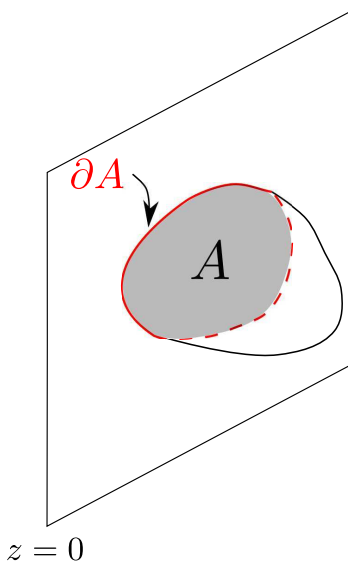


Fig. 4: The Ryu-Takayanagi conjecture. The entanglement entropy of a region A in a conformal field theory is given by the area of a minimal surface in the bulk of AdS that ends on ∂A (the boundary of region A) at the boundary of AdS .

In principle, we can compute the entanglement entropy of the region A by using the replica trick [28,29]. This is a general method for computing entanglement entropy in quantum field theories. The idea is to take n copies of the field theory and match them together so that by moving in a circle around ∂A we go from one copy to the next. Going n times around this circle we come back to the original copy. Thus at ∂A there is a conical defect with a $2\pi n$ opening angle. This appears to be a singular metric. However, one can choose a conformal factor that diverges at ∂A in such a way that the size of the circle around ∂A is finite.

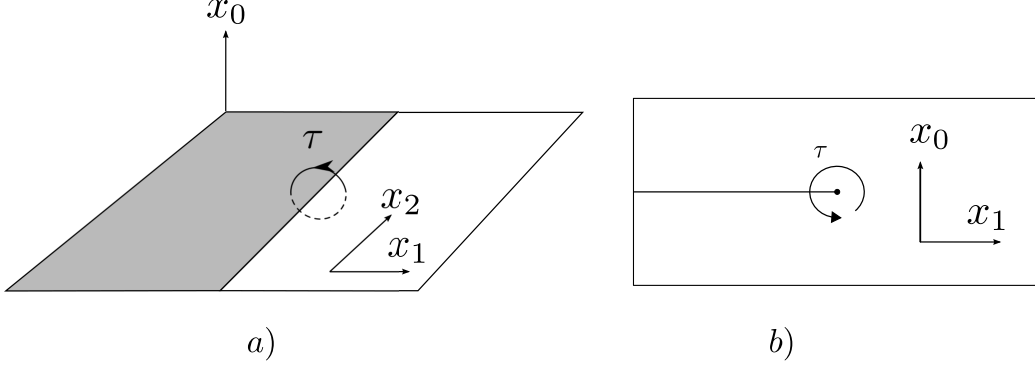


Fig. 5: Geometries that we are considering to compute the entanglement entropy in the field theory. a) Semiplane. Region A is half of the plane and its boundary, ∂A , is at $x_0 = x_1 = 0$. b) x_0, x_1 view of the semiplane and the coordinate τ .

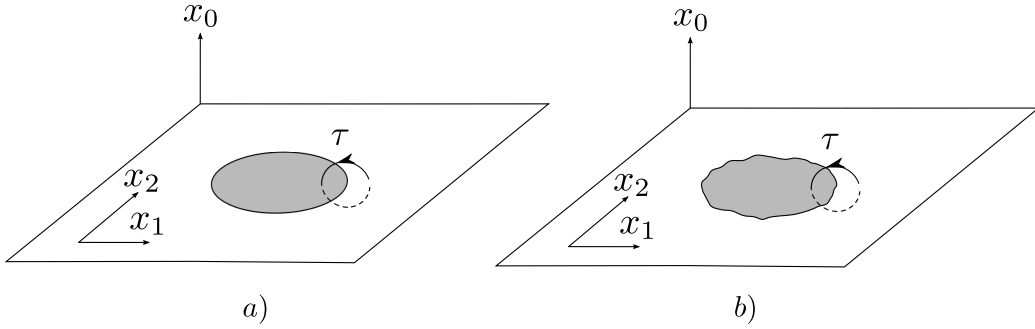


Fig. 6: Other geometries in three dimensions. a) Disk configuration. b) Slightly deformed disk. The τ coordinate goes around ∂A .

This is most easily understood for simple regions [30,31]. Imagine we have a conformal field theory $R^{1,d-1}$ at the boundary. Then we can choose a region A defined by $x_1 > 0$, see figure fig. 5. The boundary of the region is the surface $x_0 = 0, x_1 = 0$. Going to Euclidean space, R^d , we can combine the directions x_0 and x_1 into two directions labeled in polar coordinates by r and τ . The metric is

$$ds^2 = r^2 d\tau^2 + dr^2 + d\vec{x}^2 \quad \rightarrow \quad d\tau^2 + \frac{dr^2 + d\vec{x}^2}{r^2} \quad (5.1)$$

where \vec{x} are the rest of the spatial coordinates. In the right hand side of (5.1) we have multiplied by an overall conformal factor $1/r^2$ to put the metric in the form of $S^1 \times H_{d-1}$. We can now easily perform the replica trick, it corresponds to changing the length of S^1 from 2π to $2\pi n$. Clearly this metric, $S_n^1 \times H_{d-1}$ is a perfectly legal metric and we can consider its gravity dual. It is a certain black brane. In this case, we have a $U(1)$ isometry in the rescaled coordinates and then the entropy computed using the replica trick

or using the ordinary Gibbons-Hawking formula is exactly the same. Note that at the AdS boundary the circle S^1 has a nonzero size everywhere. In the interior of AdS it shrinks to zero at a “horizon”. Notice, in particular, that the half space region we discussed above can be conformally mapped to a spherical region $\sum x_i^2 \leq 1$. In this case, the circle S^1 appearing in (5.1) corresponds to a coordinates that goes around ∂A as in figure fig. 6.

Now, this was a very simple region. If we consider more complicated regions, then it is not possible to choose a system of coordinates and a conformal rescaling such that the metric is independent of the angular direction τ . In all cases we will have an angular direction, τ , since it is the direction we used to perform the replica trick construction. The choice of this coordinate is completely arbitrary, as long as it goes around the boundary of region A . As we go near ∂A we have a problem which locally looks like (5.1), and we can choose a conformal factor which makes the metric non-singular as in (5.1) for all the replicas. The difference with (5.1) is that, as we increase r , we will have extra terms in the metric that can have some τ dependence. This dependence always involves powers of $e^{\pm i\tau}$ since this is just the statement that the τ direction is parametrizing circles in the original boundary geometry. All the statements in this paragraph involve the boundary geometry, the geometry where the field theory lives. These replica trick boundary geometries simply amount to letting the circle have size $\tau \sim \tau + 2\pi n$, *without changing any of the functions that appear in the boundary geometry*. All such functions are periodic under $\tau \rightarrow \tau + 2\pi$ ¹².

Thus, the field theory replica trick, can be translated, via the standard AdS/CFT dictionary [32,33], to a problem in gravity which is identical to the problem that we discussed in section 4. Here no conjecture is involved other than the original AdS/CFT relation. The replica trick then defines the entropy as in (1.3). In order to do that, we need to analytically continue in n to $n \sim 1$.

The Ryu-Takayanagi conjecture boils down to a statement in classical geometry. It is the statement we discussed in section 4. Computing $\log Z(n)$, using smooth geometries, analytically continuing in n , and computing the entropy defined in (1.3) gives the area formula in (1.4).

¹² If we want to explicitly parameterize the metric in this way, we might need to choose different coordinate patches, as usual. When the coordinate patches are chosen in a τ dependent fashion, then the $\tau \rightarrow \tau + 2\pi n$ identification can produces spaces with an n dependent topology. This happens, for example, in the case that we have two separate intervals in a two dimensional CFT. (We thank Xi Dong for a discussion on this.)

Notice that in a setup where A is a spatial region contained at $x^0 = 0$ on the boundary, then there is a time reflection symmetry $x^0 \rightarrow -x^0$, which translates into $\tau \rightarrow -\tau$ for the circle in the Euclidean solution. This implies that we can go to Lorentzian signature, as usual with $x^0 \rightarrow ix^0$. This translates into $\tau \rightarrow it$. Now the region where the τ circle is shrinking to zero corresponds to a horizon in the bulk. It is a horizon for an observer sitting at fixed small r .

There is a generalization of the Ryu-Takayanagi conjecture for situations that are time dependent [34]. It again involves an extremal surface ending on ∂A , but in the full Lorentzian spacetime. In those cases there is no obvious Euclidean continuation to perform the replica trick. This suggests that there should be a way to think about the problem which does not go through the Euclidean solutions and the replica trick. We should remark that in some cases we can perform a replica trick in the Euclidean geometry for regions that depend on the Euclidean time and then one can analytically continue to the Lorentzian signature solutions. Some examples were discussed in [35].

5.1. General entanglement interpretation

In the introduction, we presented the computation of the generalized gravitational entropy as a property of the density matrix constructed by integrating over a circle in Euclidean time. It is natural to ask whether there is a general Lorentzian interpretation that involves entanglement. This is indeed the case in the Ryu-Takayanagi discussion of entanglement of a subregion of the boundary.

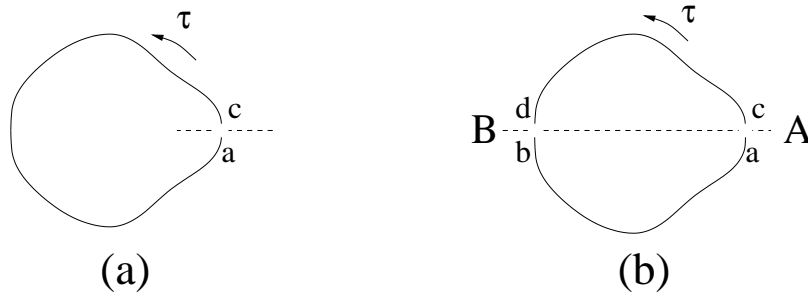


Fig. 7: We consider periodic boundary conditions with a reflection symmetry $\tau \rightarrow -\tau$. In (a) we see that by cutting at $\tau = 0$ we get a density matrix ρ_{ac} , where a and c label the states on the two sides of the cut. In (b) we note that we can cut along the moment of time reflection symmetry $\tau = 0, \pi$. Then we get a pure state in two separate Hilbert spaces labeled by A and B . The bottom half of the picture can be viewed as a state ψ_{ab} and the top part as ψ_{cd}^\dagger . Tracing out over the B Hilbert space, we recover $\rho_{ac} = \sum_b \psi_{ab} \psi_{cb}^\dagger$. At this moment of time reflection symmetry we can also continue to Lorentzian signature.

Here we would like to point out that in very general situations we can also have an entanglement interpretation. Suppose that the boundary conditions have a moment of time reflection symmetry. Say that this acts as $\tau \rightarrow -\tau$. Then by cutting the boundary conditions at $\tau = 0, \pi$ we can interpret the lower part of the evolution as specifying a pure state $|\Psi\rangle$ in the product of two theories, which we call A and B . See fig. 7. Similarly, the upper part can be viewed as specifying the state $\langle\Psi|$. The density matrix can then arise by tracing over one or the other subsystem. And the entropy can be interpreted as entanglement entropy for system A with B . This is the same as in the eternal black hole discussion [36,37].

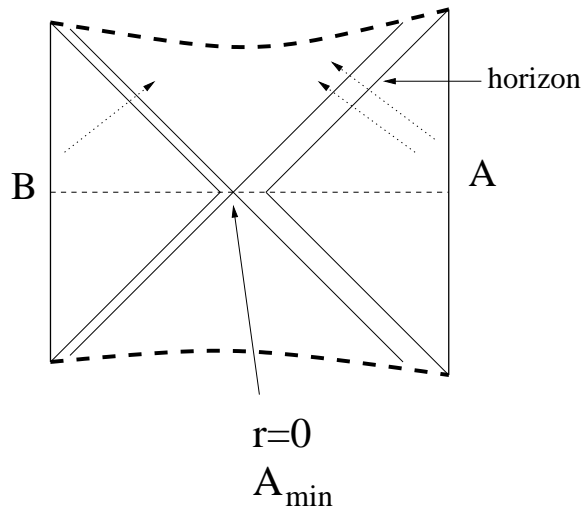


Fig. 8: Here we consider a situation with asymptotically AdS boundary conditions. The boundary conditions contain a small time dependent deformation which vanishes at infinity. So in the far future we settle down into a stationary black hole on both sides. The entropy of these black holes is bigger than the entropy of the initial entanglement since, the time dependent boundary conditions have sent in energy and have increased the entropy of the system. In other words, there was a non-zero flux of energy through the horizon which increased its area. The dotted lines indicate the matter falling through the horizon.

The bulk solution is also expected to have a time reflection symmetry in this case. Under $\tau \rightarrow it$ we get a Lorentzian solution. The vicinity of $r = 0$ looks locally like Rindler space. This procedure generically produces a time dependent solution and we might get singularities or horizons in the boundary conditions. We can consider a situation where the Lorentzian time evolution can be performed out to infinite time without ever connecting again the two boundary regions or encountering singularities on the boundary.

An example is the following. We start from a Euclidean black hole but with a small perturbation of the boundary conditions which is smooth in Euclidean time and goes to zero at large Lorentzian time. More concretely, we can consider the model of section 2 and set the boundary conditions $\phi_B = \frac{\eta(1+\cos\tau)}{2+\cos 2\tau}$. When we go to Lorentzian time this becomes $\phi_B = \frac{\eta(1\pm\cosh\tau)}{2+\cosh 2t}$ where the \pm corresponds to the A and B sides respectively. Note that these go to zero at large times. We expect that solution should be qualitatively like fig. 8. A very explicit solution with these characteristics was studied in [38,39]¹³.

In cases that arise from entanglement of subregions via AdS/CFT, the fact that the causal horizon is closer to the boundary than the minimal surface that computes the entanglement entropy was noted in [34](see also [40,41,42,43]).

6. Conclusions and discussion

In this article we have noted that we can generalize the concept of Euclidean gravitational entropy to more general situations than the ones associated to thermal equilibrium. In particular, we have considered euclidean solutions that contain a circle $\tau \rightarrow \tau + 2\pi$. We have introduced a boundary, setting boundary conditions which are τ dependent but periodic under $\tau \rightarrow \tau + 2\pi$. Thinking of gravity as a holographic theory, we view these boundary conditions as defining the system. Euclidean evolution on the circle produces an un-normalized density matrix. The Euclidean solution gives us the trace of this density matrix. By performing the gravity version of the replica trick we have defined traces of n^{th} powers of the density matrix. These are geometries with exactly the same boundary condition as functions of τ , but where the τ variable is taken to have period $\tau \rightarrow \tau + 2\pi n$. For integer n the bulk geometries are smooth and free of any conical defects. These geometries are computing the trace of the n^{th} power of the density matrix. By analytically continuing in n and taking a derivative near $n = 1$ we can compute a quantity that is interpreted as the Von-Neumann entropy of the underlying density matrix. Note that all computations are classical. The density matrix we are talking about is a hypothetical density matrix

¹³ The solutions in [38,39] are based on Janus solutions. Their boundary in Euclidean space has the form $S^1 \times \Sigma$ where Σ is a quotient of hyperbolic space. The S^1 is divided in two equal parts and the dilaton has a different value on each part. The Lorentzian continuation is obtained by continuing across the moment with a time reflection symmetry. The two boundaries different values for the dilaton. These values are constant in time. The bulk smoothly interpolates between the two.

in some underlying theory of quantum gravity. In AdS/CFT situations we can give an precise definition for this density matrix.

A version of the Ryu-Takayanagi conjecture is that this generalized gravitational entropy, computed in this fashion, is given by the area of a minimal area surface in the original geometry (the solution with $n = 1$).

We have given some arguments for the correctness of the Ryu-Takayanagi conjecture. The arguments involved the assumption that we can analytically continue the geometries away from integer values of n . We further made the assumption that these analytically continued geometries, for small $\epsilon \equiv n - 1$, are smooth in the two directions transverse to the minimal area surface but can have mild singularities which are not important for evaluating the action. We do not view these metrics as physically meaningful, we view them just as a tool for deriving the Ryu-Takayanagi formula. Our assumptions were motivated by considering a simple example, described in section 2. But we have no further justification other than the fact that they hold in this example and seem reasonable assumptions. We have derived the minimal area condition by demanding the existence of a small deviation away from the $n = 1$ solution that is consistent with our assumptions on the type of singularities that are allowed. One simple way to state the type of allowed singularities is to do a dimensional reduction of the whole configuration to the two dimensions transverse to the minimal surface. Then we have a two dimensional metric, a dilaton field that multiplies the two dimensional curvature in the action and a set of other fields. Then the metric should be smooth and the gradient of the dilaton at the minimal surface should be zero, which is the minimal area condition. All other fields can have mild singularities of the form $\phi \sim z|z|^{2\epsilon}$ at the origin. When n is not an integer we evaluate the gravitational action by integrating τ between $[0, 2\pi]$ and then multiplying by n . We have also argued that this method gives rise to the area formula for the entropy, essentially for the same reasons as for the case with the $U(1)$ symmetry. One way to understand this is that all non $U(1)$ invariant fields are going to zero at the origin. Then the methods described in section 3 give the usual formula.

An alternative way to view the solutions is to imagine that we keep the original period of the circle, $\tau \sim \tau + 2\pi$ but we introduce a cosmic string (or cosmic $D - 3$ brane) with a $2\pi/n$ opening angle. In addition, we multiply the resulting action by a factor of n . For n close to one we have a very light cosmic string that deforms the geometry very slightly. We can then view the entropy formula as arising from the Nambu action for this cosmic

string. Also the minimal area condition comes from minimizing this Nambu action. The long and detailed discussion that we presented tried to justify these statements in detail.

One interesting open question is whether one can generalize the derivation to the time dependent case considered in [34], where, generically, there is no obvious Euclidean continuation.

Another interesting direction is to generalize the discussion to gravity with higher derivatives. The most naive conjecture is that the entropy is given by the Wald formula. However, this conjecture was argued to be wrong in [44], where a modified conjecture was made for the case of Lovelock gravity. A more informed conjecture is to say that we get the Wald-Iyer formula proposed in section 7 of [21]. In fact, this reduces to the proposal in [44] for Lovelock gravity. It would be interesting to see whether this is correct and what the equations for the surface are.

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Appendix A. Example of a scalar field in AdS_3

In this appendix, we consider a massive scalar field in AdS_3 and show explicitly that the entropy that we compute using the replica trick is equal to the modification of the area due to the presence of a non-zero scalar field background.

A.1. Massive scalar field

For a massive scalar field we have equations which are very similar to the ones in the text. We consider a complex scalar field of mass m . Setting the radius of AdS_3 to one we need to impose the boundary condition

$$\phi|_{r_c} = \eta e^{i\tau} r_c^{\Delta-2}, \quad \Delta = 1 + \sqrt{m^2 + 1} \quad (\text{A.1})$$

where Δ is the scaling dimension of the corresponding operator and r_c is a large value of r which represents the cutoff surface. The relevant solution of the wave equation on the metric (2.3) is

$$\phi = \eta e^{i\tau} \frac{f(nr)}{f(nr_c)} r_c^{\Delta-2}, \quad f(r) = r^n {}_2F_1\left(\frac{n}{2} - \frac{\Delta}{2} + 1, \frac{n}{2} + \frac{\Delta}{2}; n + 1; -r^2\right) \quad (\text{A.2})$$

We then evaluate

$$\begin{aligned} \log Z(n) &= - \int d^3x \sqrt{g} [|\nabla\phi|^2 + m^2|\phi|^2] = -(2\pi n) L_x \phi_{r_c}^* r_c^3 \partial_r \phi \Big|_{r_c} = \\ &= (2\pi L_x) |\eta|^2 [B(n, \Delta) + \text{linear in } n] \end{aligned} \quad (\text{A.3})$$

where the terms linear in n also include all divergent terms. It is important that these counterterms do not give rise to any non-trivial n dependence. This is due to the fact that we keep the τ -dependence of the boundary conditions fixed as we vary n . We also defined

$$B(n, \Delta) = - \frac{2n^{3-2\Delta} \Gamma(2-\Delta) \Gamma\left(\frac{n+\Delta}{2}\right)^2}{\Gamma(\Delta-1) \Gamma\left(\frac{n-\Delta}{2} + 1\right)^2} \quad (\text{A.4})$$

We can then compute the entropy to order η^2 from (1.3), which gives

$$\begin{aligned} S|_{\eta^2} &= -n \partial_n [\log Z(n) - n \log Z(1)]|_{n=1} = \\ &= -\eta^2 \left\{ \frac{4\pi [2(\Delta-2)\Delta + (1-\Delta)\pi \tan(\pi\Delta/2)] \Gamma(2-\Delta) \Gamma\left(\frac{\Delta+1}{2}\right)^2}{\Gamma\left(\frac{3-\Delta}{2}\right)^2 \Gamma(\Delta)} \right\} \end{aligned} \quad (\text{A.5})$$

A.2. Change in the metric from Einstein's equations

Now we will study the backreaction of the scalar in the metric. The action is

$$-S = \int_{AdS_3} [R - 2\Lambda - |\nabla\phi|^2 + m^2|\phi|^2] \quad (\text{A.6})$$

with $\Lambda = -1$. The equations of motion are

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}(R-2) = T_{(\mu\nu)} \quad (\text{A.7})$$

where $T_{\mu\nu} = \partial_\mu \phi^* \partial_\nu \phi - \frac{g_{\mu\nu}}{2} (|\nabla\phi|^2 + m^2|\phi|^2)$. The ansatz for the metric is

$$ds^2 = \frac{1}{r^2 + g(r) + 1} dr^2 + (r^2 + 1) (1 + v(r)) dx^2 + r^2 dt^2 \quad (\text{A.8})$$

where $g(r), v(r)$ will be $O(\eta^2)$. If we expand Einstein equations to first order we obtain three equations for the diagonal components. There are only two independent equations since the last one will give us the scalar wave equation when the first two are satisfied

$$\begin{aligned} g'(r) &= T_{xx} \frac{2r}{r^2 + 1} \\ v'(r) &= 2r T_{rr} - \frac{2rg(r)}{(r^2 + 1)^2} \end{aligned} \quad (\text{A.9})$$

Since we consider a configuration with $\partial_x \phi = 0$, we can relate the components of the stress energy tensor: $T_{rr} = (\partial_r \phi)^2 + (1 + r^2)^{-2} T_{xx}$. We then find

$$v'(r) = 2r(\partial_r \phi)^2 + \frac{\partial}{\partial r} \left(\frac{g(r)}{r^2 + 1} \right) \quad (\text{A.10})$$

And

$$v(0) = -2 \int_0^\infty dr r |\partial_r \phi|^2 \quad \rightarrow \quad S|_{\eta^2} = 4\pi \delta A = -\eta^2 (4\pi L_x) \int_0^\infty dr r |\partial_r \phi(r)|^2 \quad (\text{A.11})$$

where we use that the second term in (A.10) is a total derivative and that $g(0) = 0$ due to the regularity condition for the metric at the origin. In addition $g/r^2 \rightarrow 0$ at infinity. In our units ($16\pi G_N = 1$), the black hole formula is $S = 4\pi A = 4\pi A_0(1 + \frac{v(0)}{2}) = 4\pi(A_0 + \delta A)$. Substituting the solution for $\phi(r)$ for $n = 1$ (A.2), and integrating, we get the same as in (A.5). We checked this only numerically, but below we will show it without performing the explicit calculation.

A.3. The two quantities are the same

In the above computation we actually did not need to solve all the equations to the end in order to show that the two results are the same.

We will rearrange the entropy formula for the scalar so that we get an expression that is simpler to compare with the area contribution. The lagrangian $\mathcal{L}(g_{\mu\nu}, \phi, \nabla_\mu \phi)$ is a function of τ . When we evaluate the gravitational action, we integrate over all coordinates except τ . Then we first integrate over τ from zero to 2π and then multiply by n . We can do this both for integer or non-integer n . We denote the τ integral as $[\log Z(n)]_{2\pi}$. Then we have

$$\log Z(n) = n[\log Z(n)]_{2\pi} \quad (\text{A.12})$$

Then the entropy formula (1.3) simplifies and we get

$$S = -n \partial_n \{n[\log Z(n)]_{2\pi} - n \log Z(1)\}|_{n=1} = -\partial_n [\log Z(n)]_{2\pi}|_{n=1} \quad (\text{A.13})$$

And the later expression can be straightforwardly evaluated, using¹⁴ $\sqrt{g} T_{\mu\nu} = \frac{\partial \sqrt{g} \mathcal{L}}{\partial g^{\mu\nu}}$,

$$\begin{aligned} -\partial_n [\log Z^{\text{matter}}(n)]_{2\pi} &= \int_0^{2\pi} d\tau \int dx dr \sqrt{g} \left(T_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} + \frac{\partial \mathcal{L}}{\partial \phi} \partial_n \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_n \partial^\mu \phi \right) = \\ &= \int_0^{2\pi} d\tau \int dx dr \sqrt{g} T_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} \end{aligned} \quad (\text{A.14})$$

¹⁴ We define it like this because the action is $\log Z(n) = \int_{AdS_3} (\mathcal{L}_{\text{Grav}} - \mathcal{L}_{\text{matter}})$ so the field equations read $G_{\mu\nu} = T_{\mu\nu}$.

In the last line we used the equations of motion (of course $\frac{\delta S}{\delta \phi} = 0$). One can check that the expression with the stress energy tensor gives us $\sqrt{g}T_{\mu\nu}\frac{\partial g^{\mu\nu}}{\partial n}|_{n=1} = -2\eta^2 r f'^2$, so

$$S - S_0 = -\eta^2 4\pi L_x \int dr r f'^2 \quad (\text{A.15})$$

In writing (A.14) we have only included the action of the scalar field in the computation.

We can now show that we get the area, without using explicit expressions. This can be done as follows. First note that in the second line of (A.14) we can use Einstein's equation to write $T_{\mu\nu}$ in terms of the Einstein tensor, which is related to the variation of the gravitational action. We end up with an expression of the form

$$\int d\tau dx dr \sqrt{g} G_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} \Big|_{n=1} \quad (\text{A.16})$$

This is closely related to the derivative of the gravitational part of the action. As we explained above we know that the gravitational part of the action has no term of order η^2 . Thus we know that the ∂_n derivative of the gravitational part vanishes at order η^2 . This derivative is the same as (A.16) up to a total derivative term

$$\partial_n [\log Z^{\text{Grav}}(n)]_{2\pi}|_{\eta^2} = 0 = 2\pi \left[\int dx dr \sqrt{g} G_{\mu\nu} \frac{\partial g^{\mu\nu}}{\partial n} - \int dx \sqrt{g} \nabla_\mu \partial_n g^{\mu r} \Big|_{r=0} \right]_{\eta^2} \quad (\text{A.17})$$

The last term gives the area of the horizon, or more precisely the area of the horizon at order η^2 .

These are the same manipulations that one can do in general, but we have done all steps explicitly above to check that everything indeed works in situations with no $U(1)$ symmetry.

A.4. Real scalar

We now consider the case of a real scalar field $\phi = f(r) \cos \tau$, $f(r)$ is the same as before but now the stress energy tensor no longer has the $U(1)$ symmetry

$$T_{\mu\nu} = T_{\mu\nu}^0 + T_{\mu\nu}^1 e^{i2\tau} + T_{\mu\nu}^{-1} e^{-i2\tau} \quad (\text{A.18})$$

And $T_1^* = T_{-1}$. The metric has the same fourier decomposition, so v and g in (A.8) also have three fourier components. The entropy coming from the change in the area is

$$S|_{\eta^2} = \int_0^{2\pi} d\tau v^0(0) = 2\pi v^0(0) \quad (\text{A.19})$$

where v^0 is the constant component of v . It is easy to check that $v^0(0) = -\int dr r f'^2$.

The scalar action contributes as follows

$$S|_{\eta^2} = \int d\tau dr (-2r f'^2 \cos^2(k\tau)) = -2\pi \int dr r f'^2 \quad (\text{A.20})$$

So we find agreement once more, and the result is precisely half of the complex scalar.

Appendix B. Derivation of minimal area condition for the general case from a explicit calculation

In this appendix, we obtain the minimal area condition of section 4 without using dimensional reduction. As in section 4, we derive this condition from requiring that the analytically continued solution satisfies the linearized equations of motion near $r = 0$.

The metric of the $n = 1$ solution, which satisfies (locally) the equations of motion is

$$\begin{aligned} ds^2 &= dx_1^2 + dx_2^2 + g_{ij}(dy^i + b_\alpha^i dx^\alpha)(dy^j + b_\alpha^j dx^\alpha) + o(r^2) , \\ g_{ij} &= h_{ij} + x_1 K_{ij}^1 + x_2 K_{ij}^2 , \quad b_\alpha^i \sim o(r) \end{aligned} \quad (\text{B.1})$$

Here, y_i are the directions along the surface. Now, we do the replica trick, that is, we change the periodicity of the τ circle from 2π to $2\pi n$ and analytically continue n to $1 + \epsilon$. In this way, the metric will be modified to linear order in ϵ

$$ds^2 = e^{2\rho}(dr^2 + r^2 d\tau^2) + g_{ij}(dy^i + b_\alpha^i dx^\alpha)(dy^j + b_\alpha^j dx^\alpha) + \delta g \quad (\text{B.2})$$

Where we decomposed the perturbation in a part that makes the metric smooth $\rho = \delta\rho = -\epsilon \log r$ and a perturbation δg that has components δg_{ab} valued in all directions. For simplicity we work with z, \bar{z} coordinates: $x_1 = \frac{z+\bar{z}}{2}, x_2 = \frac{z-\bar{z}}{2i}$. As a gauge condition, we set $\delta g_{zz} = \delta g_{\bar{z}\bar{z}} = 0$. We also set $\delta g_{z\bar{z}} = 0$, since this variation is included in ρ . We require the perturbation, δg_{ab} to be periodic: $\delta g_{ab}(\tau) \sim \delta g_{ab}(\tau + 2\pi)$.

We want to compute the linearized equation of motion $\delta G_{zz} = \delta T_{zz}$. In particular, we want to focus on the terms that can be divergent, going like $1/r$ near the origin. We find

$$\begin{aligned} \delta R_{zz} &= \frac{-\epsilon}{z} K_z + \frac{1}{2}(2\delta g^p{}_{z;zp} - \delta g_{;zz} - \nabla^2 \delta g_{zz}) + (\text{regular as } r \rightarrow 0) \\ &= \frac{-\epsilon}{z} K_z - \frac{1}{2} \partial_z^2 \delta\gamma + \dots \end{aligned} \quad (\text{B.3})$$

where $\delta\gamma \equiv g^{ij} \delta g_{ij}$ and $K_z = \frac{K^1 - iK^2}{2}$. In (B.3) we neglected the terms that have y_i derivatives because we expect them to be regular, only terms with two x^α derivatives can contribute to this order.

Now, since the stress energy tensor is not expected to be singular, the equations of motion imply that the two potentially divergent terms should cancel

$$\begin{aligned} \frac{1}{2} \partial_z^2 \delta\gamma &= \frac{-\epsilon}{z} K_z \\ \frac{1}{2} \partial_{\bar{z}}^2 \delta\gamma &= \frac{-\epsilon}{\bar{z}} K_{\bar{z}} \end{aligned} \quad (\text{B.4})$$

These are the same equations as before (4.7), which are only satisfied for a periodic function, $\delta\gamma(\tau) \sim \delta\gamma(\tau + 2\pi)$, if $K_z = K_{\bar{z}} = 0$. Note that although the equations of motion are well behaved for $K_z = 0$, the Riemann tensor diverges, as we discussed in section 2. This discussion is similar to the analysis in [26] for the motion of a cosmic string.

Appendix C. Computation of the entropy for a disk

Here we consider a very simple example of gravitational entropy. We go through it to explain how one can put boundary conditions at fixed distance.

Consider the metric $ds^2 = dr^2 + r^2 d\tau^2$. In addition, we can have other dimensions, but let us assume we can ignore them. In this case, we can say that we pick an $r = r_c$ and we set up the boundary conditions there. We demand that the metric in the angular direction is

$$ds_{bdy}^2 = r_c^2 d\tau^2 \tag{C.1}$$

at the boundary $r = r_c$. We now consider the situation with $\tau \sim \tau + 2\pi n$. We should consider now metrics with the same boundary condition (C.1), but compatible with the new period. These metrics are

$$ds^2 = n^2 dr^2 + r^2 d\tau^2 \tag{C.2}$$

We can evaluate the gravitational action for these spaces and obtain

$$\log Z(n) = \frac{1}{16\pi G_N} \left[\int \sqrt{g} R + 2 \int_{bdy} K \right] = \frac{A}{4G_N} \tag{C.3}$$

which is independent of n . Here A is the area of the transverse directions which were not explicitly mentioned above. Using the usual formula, we get the expected area formula for the entropy.

We have included this trivial computation to explicitly show how gravity regularizes the divergent contribution that one normally gets in field theory. In fact, there is no divergence because there was no conical space in this computation!. Of course, this begs the question of whether the finite part of the one loop correction computed by performing a one loop computation around the above geometries is indeed the same as the finite part of the one loop corrections computed using the conical spaces that appear in the field theory discussion of the replica trick.

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