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Relative Entropy of States of von Neumann Algebras

By

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Abstract

Relative entropy of two states of a von Neumann algebra is defined in terms of the relative modular operator. The strict positivity, lower semi-continuity, convexity and monotonicity of relative entropy are proved. The Wigner-Yanase-Dyson-Lieb concavity is also proved for general von Neumann algebra.

§ 1. Introduction

A relative entropy (also called relative information, see [12], [14]) is a useful tool in the study of equilibrium states of lattice systems ([2], [4], [6]). For normal faithful positive linear functionals ϕ and ψ of a von Neumann algebra $\mathfrak{M}$, the relative entropy is defined by

$$S(\phi/\psi) \equiv -(\Psi, (\log \Delta_{\Phi,\Psi})\Psi) \tag{1.1}$$

where $\Delta_{\Phi,\Psi}$ is the relative modular operator of cyclic and separating vector representatives Φ and Ψ of ϕ and ψ, and (1.1) is independent of the choice of vector representatives Φ and Ψ. The definition (1.1) coincides with usual definition

$$S(\rho_\phi/\rho_\psi) = \mathrm{tr}\,(\rho_\psi \log \rho_\psi) - \mathrm{tr}\,(\rho_\psi \log \rho_\phi) \tag{1.2}$$

when $\mathfrak{M}$ is finite dimensional and ρ_ϕ and ρ_ψ are density matrices for ϕ and ψ.

We shall prove the following properties of $S(\phi/\psi)$.

(1) *Strict positivity*: If $\phi(\mathbf{1}) = \psi(\mathbf{1})$, then

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(1.3) $$S(\phi/\psi)\geqq 0$$

and the equality holds if and only if $\phi=\psi$.

(2) *Lower semi-continuity*: If $\lim_n \|\phi_n-\phi\|=\lim_n \|\psi_n-\psi\|=0$,

(1.4) $$\varliminf S(\phi_n/\psi_n)\geqq S(\phi/\psi).$$

(3) *Convexity*: $S(\phi/\psi)$ is jointly convex in ϕ and ψ. Namely

(1.5) $$\Sigma\lambda_i S(\phi_i/\psi_i)\geqq S(\Sigma\lambda_i\phi_i/\Sigma\lambda_i\psi_i)$$

if $\lambda_i\geqq 0$ and $\Sigma\lambda_i=1$.

(4) *Monotonicity*:

(1.6) $$S(E_{\mathfrak{N}}\phi/E_{\mathfrak{N}}\psi)\leqq S(\phi/\psi)$$

where $E_{\mathfrak{N}}\phi$ and $E_{\mathfrak{N}}\psi$ denote the restrictions of ϕ and ψ to a von Neumann subalgebra $\mathfrak{N}$ of $\mathfrak{M}$, and $\mathfrak{N}$ is assumed to be one of the following:

(Case α) $\mathfrak{N}=\mathfrak{A}'\cap\mathfrak{M}$ for a finite dimensional abelian von Neumann subalgebra $\mathfrak{A}$ of $\mathfrak{M}$.

(Case β) $\mathfrak{M}=\mathfrak{N}\otimes\mathfrak{N}_1$.

(Case γ) $\mathfrak{N}$ is approximately finite (i.e. generated by an increasing net of finite dimensional subalgebras). This case includes any finite dimensional $\mathfrak{N}$.

In the proof of convexity, we prove that

(1.7) $$\|(\Delta_{\Phi,\Psi})^{p/2}x\Psi\|^2$$

is jointly concave in ϕ and ψ for fixed $x\in\mathfrak{M}$ and $p\in[0, 1]$. (Wigner-Yanase-Dyson-Lieb concavity.)

For connection of these general results with finite matrix inequalities, see [7].

§2. Strict Positivity and Lower Semi-Continuity

We shall take Φ and Ψ to be unique vector representatives of ϕ and ψ in a fixed natural positive cone $V=V_\Psi=V_\Phi$ ([3]). Then

(2.1) $$\Phi=(\Delta_{\Phi,\Psi})^{1/2}\Psi.$$

Let E_λ be the spectral projections of $\Delta_{\Phi,\Psi}$. Then

$$S(\phi/\psi) = -\int_0^\infty \log\lambda \, d(\Psi, E_\lambda\Psi). \tag{2.2}$$

By (2.1),

$$\int_0^\infty \lambda \, d(\Psi, E_\lambda\Psi) = \phi(\mathbb{1}) < \infty. \tag{2.3}$$

Hence (2.2) is definite and gives either real number or $+\infty$.

Since the numerical function $\log\alpha$ is concave,

$$\int_0^\infty \log\alpha(\lambda) \, d\mu(\lambda) \leqq \log\int_0^\infty \alpha(\lambda)\, d\mu(\lambda) \tag{2.4}$$

for any positive measurable function $\alpha(\lambda)$ of $\lambda \in (0, \infty)$ and any probability measure μ on $(0, \infty)$. By taking $\alpha(\lambda) = \lambda^{1/2}$ and $d\mu(\lambda) = d(\Psi, E_\lambda\Psi)/\|\Psi\|^2$, the inequality (2.4) with $\log\alpha(\lambda) = (\log\lambda)/2$ yields

$$S(\phi/\psi) \geqq -2\psi(\mathbb{1})\log\{(\Phi, \Psi)/\psi(\mathbb{1})\}. \tag{2.5}$$

By Schwartz inequality,

$$(\Phi, \Psi) \leqq \|\Phi\| \, \|\Psi\| = (\phi(\mathbb{1})\psi(\mathbb{1}))^{1/2}. \tag{2.6}$$

Hence the right-hand side of (2.5) is non-negative when $\phi(\mathbb{1}) = \psi(\mathbb{1})$ and the equality holds only if the equality holds in (2.6), namely only if $\Phi = \Psi$. This proves the strict positivity. (An alternative proof follows from $\log\lambda \leqq \lambda - 1$.)

To prove lower semicontinuity, let ϕ_n, ϕ, ψ_n and ψ be normal faithful positive linear functionals of $\mathfrak{M}$ such that

$$\lim_n \|\phi_n - \phi\| = 0, \qquad \lim_n \|\psi_n - \psi\| = 0. \tag{2.7}$$

Let Φ_n, Φ, Ψ_n and Ψ be vector representatives of ϕ_n, ϕ, ψ_n and ψ in V. Then

$$\lim_n \|\Phi_n - \Phi\| = 0, \qquad \lim_n \|\Psi_n - \Psi\| = 0 \tag{2.8}$$

and hence

(2.9) $$\lim_n (\mathbb{1}+\Delta_{\Phi_n,\Psi_n}^{1/2})^{-1}=(\mathbb{1}+\Delta_{\Phi,\Psi}^{1/2})^{-1}$$

strongly. (See Theorem 4(8) in [3] and Remark 2 at the end of section 4.) Hence

(2.10) $$\lim_n f(\Delta_{\Phi_n,\Psi_n})=f(\Delta_{\Phi,\Psi})$$

for any bounded continuous function f. (See [10], Lemma 2.)

Let $\mathcal{N}=3, 4,\dots$ and

(2.11) $$f_N(\lambda)=\begin{cases}\log N & \text{if } \lambda\geqq\log N,\\ -\log N & \text{if } \lambda\leqq-\log N,\\ \lambda & \text{otherwise.}\end{cases}$$

Let E_λ^n be the spectral projection of Δ_{Φ_n,Ψ_n}. Since

$$\int_0^\infty \lambda\ d(\Psi_n, E_\lambda^n\Psi_n)=\|\Phi_n\|^2=\phi_n(\mathbb{1}),$$

we have

(2.12) $$\begin{aligned}0&\leqq\int_N^\infty(\log\lambda-\log N)\ d(\Psi_n, E_\lambda^n\Psi_n)\\ &=\int_N^\infty\{\lambda^{-1}\log(\lambda/N)\}\lambda\ d(\Psi_n, E_\lambda^n\Psi_n)\\ &\leqq\phi_n(\mathbb{1})(eN)^{-1}.\end{aligned}$$

Since

(2.13) $$\int_0^{1/N}(\log\lambda+\log N)\ d(\Psi_n, E_\lambda^n\Psi_n)\leqq0,$$

we have

(2.14) $$S(\phi_n/\psi_n)\geqq-(\Psi_n, f_N(\log\Delta_{\Phi_n,\Psi_n})\Psi_n)-\phi_n(\mathbb{1})(eN)^{-1}.$$

By using (2.10) with $f(x)=f_N(\log x)$, we obtain from (2.14)

(2.15) $$\underline{\lim}\, S(\phi_n/\psi_n)\geqq-(\Psi, f_N(\log\Delta_{\Phi,\Psi})\Psi)-\phi(\mathbb{1})(eN)^{-1}.$$

Since the right-hand side of (2.15) tends to $S(\phi/\psi)$ as $\mathcal{N}\to\infty$, we have (1.4).

§3. Unitary Cocycle

We need some properties of unitary cocycle in the proof of WYDL concavity. The unitary cocycle is defined by

$$(3.1)\qquad (\mathrm{D}\phi:\mathrm{D}\psi)_t=(\Delta_{\Phi,\Psi})^{it}\Delta_\Psi^{-it}.$$

It is unitary elements of $\mathfrak{M}$ continuously depending on real parameter t and satisfying the following equations ([8], Lemmas 1.2.2, 1.2.3 and Theorem 1.2.4):

$$(3.2)\qquad (\mathrm{D}\phi_1:\mathrm{D}\phi_2)_t(\mathrm{D}\phi_2:\mathrm{D}\phi_3)_t=(\mathrm{D}\phi_1:\mathrm{D}\phi_3)_t,$$

$$(3.3)\qquad (\mathrm{D}\phi:\mathrm{D}\psi)_t=(\mathrm{D}\psi:\mathrm{D}\phi)_t^*,$$

$$(3.4)\qquad (\mathrm{D}\phi:\mathrm{D}\psi)_t\sigma_t^\psi(x)(\mathrm{D}\phi:\mathrm{D}\psi)_t^*=\sigma_t^\phi(x),$$

$$(3.5)\qquad (\mathrm{D}\phi:\mathrm{D}\psi)_s\sigma_s^\psi\{(\mathrm{D}\phi:\mathrm{D}\psi)_t\}=(\mathrm{D}\phi:\mathrm{D}\psi)_{s+t}.$$

We now start deriving some equations useful in our proof of WYDL concavity (cf. [5]).

If $\lambda\phi\leqq\psi$ with $\lambda>0$ (and only in such a case), $(\mathrm{D}\phi:\mathrm{D}\psi)_t$ has an analytic continuation in t to the strip $0\geqq\mathrm{Im}\,t\geqq-1/2$. In other words there exists an $\mathfrak{M}$-valued function $\alpha_\phi(z)$ of z in the tube region

$$(3.6)\qquad \{z;\,0\leqq\mathrm{Re}\,z\leqq1\}$$

such that $\alpha_\phi(z)$ is strongly continuous in z on (3.6), holomorphic in z in the interior of (3.6), bounded (by $\lambda^{-\mathrm{Re}\,z/2}$) and satisfies

$$(3.7)\qquad \alpha_\phi(2it)=(\mathrm{D}\phi:\mathrm{D}\psi)_t,$$

$$(3.8)\qquad \alpha_\phi(z)\Psi=(\Delta_{\Phi,\Psi})^{z/2}\Psi,$$

$$(3.9)\qquad \alpha_\phi(1)\Psi=\Phi.$$

(For later typographical convenience, we scaled t by $2i$.)

The existence of such $\alpha_\phi(z)$ is seen as follows: First define $\alpha_\phi(z)$

on a dense set $\mathfrak{M}'\Psi$ by

(3.10)
$$\alpha_\phi(z)x'\Psi = x'(\Delta_{\Phi,\Psi})^{z/2}\Psi, \qquad x' \in \mathfrak{M}'.$$

For $z=2it$,

(3.11)
$$\alpha_\phi(z)x'\Psi = (\mathrm{D}\phi : \mathrm{D}\psi)_t x'\Psi$$

and hence

(3.12)
$$\|\alpha_\phi(z)x'\Psi\| = \|x'\Psi\|.$$

If (and only if) $\lambda^2\phi \leqq \psi$ for $\lambda>0$, there exists $A\in\mathfrak{M}$ satisfying $\|A\| \leqq \lambda^{-1/2}$ and $\Phi = A\Psi$ (Theorem 12(1) of [4]). Then

$$\begin{aligned}\Delta_{\Phi,\Psi}^{it}\Phi &= \Delta_{\Phi,\Psi}^{it}A\Psi = \sigma_t^\phi(A)\Delta_{\Phi,\Psi}^{it}\Psi \\ &= \sigma_t^\phi(A)(\mathrm{D}\phi : \mathrm{D}\psi)_t\Psi = (\mathrm{D}\phi : \mathrm{D}\psi)_t\sigma_t^\psi(A)\Psi.\end{aligned}$$

Hence for $z=2it+1$,

(3.13)
$$\alpha_\phi(z)x'\Psi = (\mathrm{D}\phi : \mathrm{D}\psi)_t\sigma_t^\psi(A)x'\Psi$$

due to (2.1) and hence

(3.14)
$$\|\alpha_\phi(z)x'\Psi\| \leqq \lambda^{-1/2}\|x'\Psi\|.$$

Since $(\Delta_{\Phi,\Psi})^{z/2}\Psi$ is holomorphic in z for $\mathrm{Re}\, z\in(0, 1)$ and continuous for $\mathrm{Re}\, z\in[0, 1]$ due to $\Psi\in \mathrm{D}(\Delta_{\Phi,\Psi}^{1/2})$ (see (2.1)), we have

(3.15)
$$\begin{aligned}\|\alpha_\phi(z)\| &= \sup_{\|f\|=1,\|x'\Psi\|=1} |(f, \alpha_\phi(z)x'\Psi)| \\ &\leqq \lambda^{-\mathrm{Re}\, z/2}\end{aligned}$$

by three line theorem. The rest follows from the definition.

Since $(\Delta_{\Phi,\Psi})^{1/2}\Psi = \Phi\in V$, we have

(3.16)
$$\Phi = \alpha_\phi(1)\Psi = J\alpha_\phi(1)\Psi = j(\alpha_\phi(1))\Psi.$$

where J is the modular conjugation operator common to vectors in V.

The analytic continuation of the cocycle equation (3.5) yields

(3.17)
$$\alpha_\phi(2is)\sigma_s^\psi\{\alpha_\phi(z)\} = \alpha_\phi(z+2is)$$

for real s and any z in (3.6). In particular

$$\alpha_\phi(1+i\theta)^*\alpha_\phi(1+i\theta)=\sigma^\psi_{\theta/2}\{\alpha_\phi(1)^*\alpha_\phi(1)\}. \tag{3.18}$$

The cocycle equation (3.5) can be rewritten as

$$(\mathrm{D}\phi:\mathrm{D}\psi)_s=(\mathrm{D}\phi:\mathrm{D}\psi)_{s+t}\sigma^\psi_s\{(\mathrm{D}\phi:\mathrm{D}\psi)^*_t\}. \tag{3.19}$$

When we apply this on Ψ, the resulting equation has the following analytic continuation:

$$\alpha_\phi(z_1)\Psi=\alpha_\phi(z_1+z_2)\Delta_\Psi^{\bar{z}_1/2}\alpha_\phi(-\bar{z}_2)^*\Psi, \tag{3.20}$$

which reduces to (3.19) (applied on Ψ) when z_1 and z_2 are pure imaginary and hence holds when z_1, $-\bar{z}_2$ and z_1+z_2 are all in (3.6). If we set $z_1=1$ and $z_2=z-1$ with $0\leqq \mathrm{Re}\, z\leqq 1$, we obtain

$$\begin{aligned}\Phi=\alpha_\phi(1)\Psi&=\alpha_\phi(z)\Delta_\Psi^{1/2}\alpha_\phi(1-\bar{z})^*\Psi\\&=\alpha_\phi(z)j(\alpha_\phi(1-\bar{z}))\Psi,\end{aligned} \tag{3.21}$$

where $j(x)=JxJ\in\mathfrak{M}'$ for $x\in\mathfrak{M}$ and $j(x)\Psi=\Delta_\Psi^{1/2}x^*\Psi$.

By the intertwining property (3.4),

$$\alpha_\phi(z)\sigma^\psi_{-iz/2}(x)=\sigma^\phi_{-iz/2}(x)\alpha_\phi(z) \tag{3.22}$$

holds for $z=2it$ and hence

$$\begin{aligned}&\alpha_\phi(z)j(\alpha_\phi(1-\bar{z}))\Delta_\Psi^{z/2}x\Psi\\&=j(\alpha_\phi(1-\bar{z}))\alpha_\phi(z)\sigma^\psi_{-iz/2}(x)\Psi\\&=j(\alpha_\phi(1-\bar{z}))\sigma^\phi_{-iz/2}(x)\alpha_\phi(z)\Psi\\&=\sigma^\phi_{-iz/2}(x)\alpha_\phi(z)j(\alpha_\phi(1-\bar{z}))\Psi\\&=\sigma^\phi_{-iz/2}(x)\Phi=\Delta_\Phi^{z/2}x\Phi,\end{aligned} \tag{3.23}$$

where (3.21) is used. Since two extreme ends of this equation have analytic continuations in z in (3.6), the equation holds for such z. In particular, for $0\leqq p\leqq 1$,

$$\alpha_\phi(p)j(\alpha_\phi(1-p))\Delta_\Psi^{p/2}x\Psi=\Delta_\Phi^{p/2}x\Phi. \tag{3.24}$$

If ϕ and χ are normal faithful positive linear functionals and

$$\psi=\lambda\phi+(1-\lambda)\chi \tag{3.25}$$

with $0<\lambda<1$, then $\psi\geqq\lambda\phi$, $\psi\geqq(1-\lambda)\chi$ with $\lambda>0$ and $1-\lambda>0$. By (3.16), we have

$$\phi(x)=(\Phi, x\Phi)=(\Psi, xj(\alpha_\phi(1)^*\alpha_\phi(1))\Psi) \tag{3.26}$$

for $x\in\mathfrak{M}$. Similarly

$$\chi(x)=(\Psi, xj(\alpha_\chi(1)^*\alpha_\chi(1))\Psi).$$

Due to (3.25), we have

$$(x^*\Psi, J\{\mathbb{1}-\lambda\alpha_\phi(1)^*\alpha_\phi(1)-(1-\lambda)\alpha_\chi(1)^*\alpha_\chi(1)\}\Psi)=0.$$

Since $x^*\Psi$, $x\in\mathfrak{M}$ are dense, $J^2=\mathbb{1}$ and Ψ is separating for $\mathfrak{M}$,

$$\mathbb{1}=\lambda\alpha_\phi(1)^*\alpha_\phi(1)+(1-\lambda)\alpha_\chi(1)^*\alpha_\chi(1). \tag{3.27}$$

If we use (3.18), we also obtain

$$\lambda\alpha_\phi(1+i\theta)^*\alpha_\phi(1+i\theta)+(1-\lambda)\alpha_\chi(1+i\theta)^*\alpha_\chi(1+i\theta)=\mathbb{1}. \tag{3.28}$$

§4. WYDL Concavity and the Convexity of Relative Entropy

First we prove the concavity of

$$f_p(\phi, x)\equiv\|\Delta_\Phi^{p/2}x\Phi\|^2 \tag{4.1}$$

in ϕ for any fixed $x\in\mathfrak{M}$ and $p\in[0, 1]$. We use the proof technique of Lieb ([11], Theorem 1).

Let ϕ, χ, λ and ψ be as in the previous section. Our aim is to prove

$$\lambda f_p(\phi, x)+(1-\lambda)f_p(\chi, x)\leqq f_p(\psi, x). \tag{4.2}$$

Consider

$$g(z)=\lambda T_\phi(z)+(1-\lambda)T_\chi(z), \tag{4.3}$$

$$T_\phi(z)\equiv(\alpha_\phi(\bar{z})j(\alpha_\phi(1-z))\Delta_\Psi^{p/2}x\Psi, \alpha_\phi(z)j(\alpha_\phi(1-\bar{z}))\Delta_\Psi^{p/2}x\Psi). \tag{4.4}$$

Since $g(z)$ is holomorphic in z on (3.6), we have

$$|g(p)| \leqq \max\{\sup_{\theta} |g(i\theta)|, \sup_{\theta} |g(1+i\theta)|\}. \tag{4.5}$$

By (3.24),

$$g(p) = \lambda f_p(\phi, x) + (1-\lambda) f_p(\chi, x). \tag{4.6}$$

By elementary inequalities,

$$|T_\phi(i\theta)| \leqq (1/2)\{\|\alpha_\phi(-i\theta) j(\alpha_\phi(1-i\theta)) \Delta_\Psi^{p/2} x\Psi\|^2 + \|\alpha_\phi(i\theta) j(\alpha_\phi(1+i\theta)) \Delta_\Psi^{p/2} x\Psi\|^2\}.$$

By the unitarity of $\alpha_\phi(i\theta)$ and by (3.28), we have

$$\lambda\|\alpha_\phi(i\theta) j(\alpha_\phi(1+i\theta)) \Delta_\Psi^{p/2} x\Psi\|^2 + (1-\lambda)\|\alpha_\chi(i\theta) j(\alpha_\chi(1+i\theta)) \Delta_\Psi^{p/2} x\Psi\|^2 = \|\Delta_\Psi^{p/2} x\Psi\|^2.$$

The other term is obtained by substitution of $-\theta$ into θ. Hence

$$|g(i\theta)| \leqq \|\Delta_\Psi^{p/2} x\Psi\|^2 = f_p(\psi, x). \tag{4.7}$$

A similar calculation starting from

$$|T_\phi(1+i\theta)| \leqq (1/2)\{\| j(\alpha_\phi(-i\theta)) \alpha_\phi(1-i\theta) \Delta_\Psi^{p/2} x\Psi\|^2 + \| j(\alpha_\phi(i\theta)) \alpha_\phi(1+i\theta) \Delta_\Psi^{p/2} x\Psi\|^2\}$$

yields

$$|g(1+i\theta)| \leqq f_p(\psi, x). \tag{4.8}$$

Collecting (4.5), (4.6), (4.7) and (4.8) together, we obtain (4.2).

Next we prove the WYDL concavity. The passage from (4.1) to

$$f_p(\phi_1, \phi_2, x) \equiv \|(\Delta_{\Phi_1,\Phi_2})^p x\Phi_2\|^2 \tag{4.9}$$

is by the 2×2 matrix trick ([8], Lemma 1.2.2).

Let $\mathfrak{M}_2$ be a 2×2 full matrix algebra with a matrix unit u_{ij} ($i=1, 2$; $j=1, 2$) acting on a 4-dimensional space $\mathfrak{K}$ with an orthonormal basis e_{ij} ($i=1, 2; j=1, 2$) satisfying $u_{ij}e_{kl} = \delta_{jk}e_{il}$. We consider the von Neumann algebra $\mathfrak{M}\otimes\mathfrak{M}_2$ acting on $\mathfrak{H}\otimes\mathfrak{K}$ instead of $\mathfrak{M}$ acting on $\mathfrak{H}$. Let

$$\hat{\Phi}=\Phi_1\otimes e_{11}+\Phi_2\otimes e_{22}, \tag{4.10}$$

where Φ_1 and Φ_2 are cyclic and separating vectors in a natural cone in $\mathfrak{H}$ corresponding to functionals $\phi_i(x)=(\Phi_i, x\Phi_i)$, $x\in\mathfrak{M}$. The vector $\hat{\Phi}$ is cyclic and separating and its modular operator yields the relative modular operator through the relation

$$(\Delta_{\hat{\Phi}})^{p/2}(x\otimes u_{12})\hat{\Phi}=\{(\Delta_{\Phi_1,\Phi_2})^{p/2}x\Phi_2\}\otimes e_{12} \tag{4.11}$$

where $x\in\mathfrak{M}$. Since

$$\hat{\phi}(\hat{x})\equiv(\hat{\Phi}, \hat{x}\hat{\Phi})=\phi_1(x_{11})+\phi_2(x_{22}) \tag{4.12}$$

for

$$\hat{x}=\Sigma x_{ij}\otimes u_{ij}, \tag{4.13}$$

$\hat{\phi}$ is linear in (ϕ_1, ϕ_2). Hence the concavity of

$$\|(\Delta_{\hat{\Phi}})^{p/2}(x\otimes u_{12})\hat{\Phi}\|^2=\|(\Delta_{\Phi_1,\Phi_2})^{p/2}x\Phi_2\|^2 \tag{4.14}$$

in $\hat{\phi}$ implies the WYDL concavity.

Let E_λ be the spectral projection of $\Delta_{\Phi,\Psi}$. The WYDL concavity just proved implies that

$$s_p(\phi/\psi)\equiv\int_0^\infty\lambda^p\,\mathrm{d}(\Psi, E_\lambda\Psi) \tag{4.15}$$

is concave jointly in ϕ and ψ, for fixed $p\in[0, 1]$. If we prove

$$S(\phi/\psi)=\lim_{p\to+0}p^{-1}\{\psi(\mathbb{1})-s_p(\phi/\psi)\}, \tag{4.16}$$

the convexity of relative entropy follows.

To prove (4.16), we note that

$$\lim_{p\to+0}p^{-1}\int_\varepsilon^\infty(1-\lambda^p)\mathrm{d}(\Psi, E_\lambda\Psi)=-\int_\varepsilon^\infty\log\lambda\,\mathrm{d}(\Psi, E_\lambda\Psi) \tag{4.17}$$

due to (2.3) and

$$p^{-1}|\lambda^p-1-p\log\lambda|\leqq(p/2)\lambda^p(\log\lambda)^2$$

for $\lambda \geqq 1$ and $p>0$. Since $1-\lambda^p \geqq 0$ for $\lambda \leqq 1$, (4.17) is a lower bound for the inferior limit of $p^{-1}\{\psi(\mathbb{1})-s_p(\phi/\psi)\}$ for $\varepsilon \leqq 1$. Hence (4.16) holds if $S(\phi/\psi)=\infty$. Since

$$0 \leqq p^{-1}(1-\lambda^p) \leqq -\log \lambda$$

for $0<\lambda \leqq 1$ and $p>0$,

$$p^{-1}\int_0^\varepsilon (1-\lambda^p)\mathrm{d}(\Psi, E_\lambda \Psi) \leqq -\int_0^\varepsilon \log \lambda \mathrm{d}(\Psi, E_\lambda \Psi)$$

tends to 0 as $\varepsilon \to 0$ uniformly in p if $S(\phi/\psi)<\infty$. Hence (4.17) implies (4.16) also for this case.

Remark 1. As a special case of WYDL concavity with $p=1/2$, we have a result of Woronowicz [15] that

$$(4.18) \qquad (\Phi, xj(x)\Psi)=(Jx^*\Phi, x\Psi)$$
$$=(\Delta_{\Phi,\Psi}^{1/2}x\Psi, x\Psi)=\|\Delta_{\Phi,\Psi}^{1/4}x\Psi\|^2$$

is concave jointly in ϕ and ψ. For $x=1$, it implies the concavity of (Φ, Ψ) in (ϕ, ψ). This implies the concavity of $\phi \to \xi(\phi)=\Phi$ in the sense that

$$(4.19) \qquad \xi(\lambda\phi_1+(1-\lambda)\phi_2)-\lambda\xi(\phi_1)-(1-\lambda)\xi(\phi_2) \in V$$

because the set of $\xi(\psi)=\Psi$ is V and V is selfdual.

Remark 2. If (2.7) and hence (2.8) hold, then

$$(4.20) \qquad \lim \|\hat{\Phi}_n-\hat{\Phi}\|=0$$

where $\hat{\Phi}_n$ and $\hat{\Phi}$ are defined by equation (4.10) where Φ_1 is replaced by Φ_n or Φ and Φ_2 is replaced by Ψ_n or Ψ. By the proof of Theorem 10 in [3],

$$(4.21) \qquad \lim_n (\mathbb{1}+\Delta_{\hat{\Phi}_n}^{1/2})^{-1}=(\mathbb{1}+\Delta_{\hat{\Phi}}^{1/2})^{-1}.$$

The subspace $\mathfrak{H}\otimes e_{12}$ of $\mathfrak{H}\otimes\mathfrak{K}$ is invariant under $(\mathbb{1}+\Delta_{\hat{\Phi}_n}^{1/2})^{-1}$ and $(\mathbb{1}+\Delta_{\hat{\Phi}}^{1/2})^{-1}$ and their restrictions to this space are

$$(\mathbb{1}+\Delta_{\hat{\Phi}}^{1/2})^{-1}(f\otimes e_{12})=\{(\mathbb{1}+\Delta_{\Phi,\Psi}^{1/2})f\}\otimes e_{12},$$

$$(\mathbb{1}+\Delta_{\hat{\Phi}_n}^{1/2})^{-1}(f\otimes e_{12})=\{(\mathbb{1}+\Delta_{\Phi_n,\Psi_n}^{1/2})f\}\otimes e_{12}.$$

Hence (2.9) holds.

Remark 3. From the 2×2 matrix method above, we can derive the following useful formula. Let $\lambda\phi_1\geqq\phi_2$ for some $\lambda\geqq 0$. In this case there exists $A\in\mathfrak{M}$ such that $\sigma_t^{\phi_1}(A)$ has an analytic continuation for $0\leqq \operatorname{Im} t\leqq 1/2$ with $\sigma_{i/4}^{\phi_1}(A)\geqq 0$, $\|A\|\leqq\lambda^{1/2}$ and

$$\phi_2(x)=\phi_1(A^*xA) \tag{4.22}$$

due to Theorem 12(1) and Theorem 14(5) of [3]. (The analyticity and positivity condition are equivalent to $A\Phi_1\in V$.) We can then prove the formula

$$\sigma_{i/2}^{\hat{\phi}}(u_{12})=A^*u_{12} \tag{4.23}$$

as follows.

Let Φ_1, Φ_2, $\hat{\Phi}$ be constructed as before. Let $\hat{J}$ be the modular conjugation operator for $\hat{\Phi}$. Then $\hat{J}(f\otimes e_{ij})=Jf\otimes e_{ji}$ (for example by Lemma 6.1 of [1]). Since $\hat{J}\Delta_{\hat{\Phi}}\hat{J}=\Delta_{\hat{\Phi}}^{-1}$, we have

$$\Delta_{\Phi_1,\Phi_2}^{-1/2}=J\Delta_{\Phi_2,\Phi_1}^{1/2}J. \tag{4.24}$$

Hence

$$\begin{aligned}\Delta_{\Phi_1,\Phi_2}^{-1/2}\Phi_2&=J\Delta_{\Phi_2,\Phi_1}^{1/2}\Phi_2=J\Delta_{\Phi_2,\Phi_1}^{1/2}A\Phi_1\\&=A^*\Phi_2,\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}\Delta_{\hat{\Phi}}^{-1/2}u_{12}\hat{\Phi}&=(\Delta_{\Phi_1,\Phi_2}^{-1/2}\Phi_2)\otimes e_{12}\\&=A^*\Phi_2\otimes e_{12}=A^*u_{12}\hat{\Phi}.\end{aligned} \tag{4.26}$$

This implies that $\sigma_t^{\hat{\phi}}(u_{12})$ has an analytic continuation $\sigma_z^{\hat{\phi}}(u_{12})\in\mathfrak{M}$ for $0\leqq \operatorname{Im} z\leqq 1/2$ satisfying

(4.27) $$\sigma_z^{\hat{\phi}}(u_{12})y'\hat{\Phi}=y'\Delta_{\hat{\Phi}}^{iz}u_{12}\hat{\Phi}, \qquad y'\in\mathfrak{M}$$

and (4.23) by Lemma 6 of [3].

§5. Some Continuity of Relative Entropy

We need the monotonicity of $(\mathbb{1}+\Delta_{\Phi,\Psi})^{-1}$ in ϕ:

Lemma 1. *If* $\lambda_1\phi_1\geqq\lambda_2\phi_2$ *for* $\lambda_1>0$, $\lambda_2>0$, *then*

(5.1) $$(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{-1}\leqq(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{-1}$$

for any $\lambda>0$.

Proof. For $x\in\mathfrak{M}$, we have

(5.2) $$\|(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{1/2}x\Psi\|^2-\|(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{1/2}x\Psi\|^2$$
$$=\lambda_1\phi_1(xx^*)-\lambda_2\phi_2(xx^*)\geqq 0,$$

where we have used

$$\|(\lambda+\lambda_j\Delta_{\Phi,\Psi})^{1/2}x\Psi\|^2=\int(\lambda+\lambda_j t)\mathrm{d}(x\Psi,\, E_t x\Psi)$$
$$=\lambda\|x\Psi\|^2+\lambda_j\|\Delta_{\Phi,\Psi}^{1/2}x\Psi\|^2=\lambda\|x\Psi\|^2+\lambda_j\|x^*\Phi\|^2$$

for $\Delta_{\Phi,\Psi}=\int t\mathrm{d}E_t$. Since $\mathfrak{M}\Psi$ is the core of $\Delta_{\Phi_1,\Psi}^{1/2}$, (5.2) implies that the domain of $(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{1/2}$ is contained in that of $(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{1/2}$ and

$$\|(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{1/2}f\|^2\geqq\|(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{1/2}f\|^2$$

for all f in the domain of $(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{1/2}$. For any $g\in\mathfrak{H}$, we take $f=(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{-1/2}g$ and we find

$$\|(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{1/2}(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{-1/2}g\|\leqq\|g\|\,.$$

Hence

$$A\equiv(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{1/2}(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{-1/2}$$

satisfies $\|A\|\leqq 1$. For $f=(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{-1/2}h$ with any $h\in\mathfrak{H}$, we have

$$\|(\lambda+\lambda_2\Delta_{\Phi_2,\Psi})^{-1/2}h\|^2=\|f\|^2\geqq\|A^*f\|^2=\|(\lambda+\lambda_1\Delta_{\Phi_1,\Psi})^{-1/2}h\|^2$$

which proves (5.1).

Lemma 2. *For* $\varepsilon>0$, *let*

$$\phi_\varepsilon=\phi+\varepsilon\psi,\qquad \psi_\varepsilon=\psi+\varepsilon\phi. \tag{5.3}$$

Then

$$\lim_{\varepsilon\to+0}\lim_{\eta\to+0}S(\phi_\varepsilon/\psi_\eta)=S(\phi/\psi). \tag{5.4}$$

Proof. First we prove

$$\lim_{\eta\to+0}S(\phi_\varepsilon/\psi_\eta)=S(\phi_\varepsilon/\psi). \tag{5.5}$$

For this, we use the formula

$$J\Delta_{\Psi,\Phi}^{-1}J=\Delta_{\Phi,\Psi}. \tag{5.6}$$

Since

$$\psi_\eta\leqq\varepsilon^{-1}\phi_\varepsilon \tag{5.7}$$

for $\varepsilon\eta<1$, there exists $A_\eta\in\mathfrak{M}$ satisfying $\|A_\eta\|\leqq\varepsilon^{-1/2}$ and

$$\Psi_\eta=A_\eta\Phi_\varepsilon\in V. \tag{5.8}$$

(Theorem 12(1) in [3].) Since $\lim\Psi_\eta=\Psi$, we have $\lim A_\eta=A_0$ where $A_0\Phi_\varepsilon=\Psi$, $\|A_0\|\leqq\varepsilon^{-1/2}$. By (5.6), we see that Ψ_η is in the domain of $\Delta_{\Phi_\varepsilon,\Psi_\eta}^{-1/2}$ and

$$\Delta_{\Phi_\varepsilon,\Psi_\eta}^{-1/2}\Psi_\eta=J\Delta_{\Psi_\eta,\Phi_\varepsilon}^{1/2}A_\eta\Phi_\varepsilon=A_\eta^*\Psi_\eta. \tag{5.9}$$

In exactly same way as the proof of the lower semicontinuity (see (2.9), (2.10), (2.11) and (2.12)), we have

$$\lim_{\eta\to+0}(\Psi_\eta, f_N(\log\Delta_{\Phi_\varepsilon,\Psi_\eta})\Psi_\eta)=(\Psi, f_N(\log\Delta_{\Phi_\varepsilon,\Psi})\Psi), \tag{5.10}$$

$$|(\Psi_\eta, (\mathbb{1}-E_1^{\varepsilon,\eta})\{\log\Delta_{\Phi_\varepsilon,\Psi_\eta}-f_N(\log\Delta_{\Phi_\varepsilon,\Psi_\eta})\}\Psi_\eta)|\leqq\phi_\varepsilon(\mathbb{1})(eN)^{-1} \tag{5.11}$$

where $\eta\geqq0$ and $\Psi_0=\Psi$. On the other hand, (5.9) implies

(5.12) $|(\Psi_\eta, E_1^{\varepsilon,\eta}\{\log \Delta_{\Phi_\varepsilon,\Psi_\eta} - f_N(\log \Delta_{\Phi_\varepsilon,\Psi_\eta})\}\Psi_\eta)| \leqq \|A_\eta^* \Psi_\eta\|^2 (Ne)^{-1}$

due to the same estimate as in (2.12). Since $\|A_\eta\| \leqq \varepsilon^{-1/2}$ independent of η, we see that

$$(5.13) \qquad \overline{\lim_{\eta\to+0}} |(\Psi_\eta, \log \Delta_{\Phi_\varepsilon,\Psi_\eta}\Psi_\eta) - (\Psi, \log \Delta_{\Phi_\varepsilon,\Psi}\Psi)| \leqq 2\{\phi_\varepsilon(\mathbb{1}) + \varepsilon^{-1/2}\psi(\mathbb{1})\}(eN)^{-1}.$$

Since $N>1$ is arbitrary, we have (5.5).

Now we prove

$$(5.14) \qquad \lim_{\varepsilon\to+0} S(\phi_\varepsilon/\psi) = S(\phi/\psi).$$

By lower semicontinuity,

$$(5.15) \qquad \underline{\lim}\, S(\phi_\varepsilon/\psi) \geqq S(\phi/\psi).$$

(If $S(\phi/\psi)=\infty$, then (5.14) follows from (5.15).)

From the formula

$$(5.16) \qquad \int_1^N \left(\frac{1}{t+\lambda} - \frac{1}{t}\right) \mathrm{d}t = \log(1+(\lambda/N)) - \log(1+\lambda)$$

we obtain

$$(5.17) \qquad F_\Phi(N) \equiv (\Psi, \log\{\mathbb{1} + (\Delta_{\Phi,\Psi} - \mathbb{1})/N\}\Psi) - (\Psi, \log \Delta_{\Phi,\Psi}\Psi) = \int_0^{N-1} (\Psi, (t+\Delta_{\Phi,\Psi})^{-1}\Psi)\mathrm{d}t - (\log N)\|\Psi\|^2.$$

(The interchange of $\mathrm{d}t$ integration and $\mathrm{d}(\Psi, E_\lambda \Psi)$ integration is allowed for positive integrant $(t+\lambda)^{-1}$.) Since $\phi_\varepsilon \geqq \phi$, Lemma 1 implies

$$(5.18) \qquad F_{\Phi_\varepsilon}(N) \leqq F_\Phi(N).$$

Since $\|\Delta_{\Phi_\varepsilon,\Psi}^{1/2}\Psi\| = \|\Phi_\varepsilon\|$ and $\|\Delta_{\Phi,\Psi}^{1/2}\Psi\| = \|\Phi\|$ are finite, we have

$$\lim_{N\to\infty} F_{\Phi_\varepsilon}(N) = -(\Psi, \log \Delta_{\Phi_\varepsilon,\Psi}\Psi),$$

$$\lim_{N\to\infty} F_\Phi(N) = -(\Psi, \log \Delta_{\Phi,\Psi}\Psi).$$

Hence

(5.19) $$S(\phi_\varepsilon/\psi)\leqq S(\phi/\psi).$$

The inequalities (5.15) and (5.19) imply (5.14).

Remark. The above proof shows that if $\phi_1\leqq\phi_2$, then $S(\phi_1/\psi)\geqq S(\phi_2/\psi)$. The same conclusion follows also from $\Phi_2-\Phi_1\in V$.

Lemma 3. *Let $\mathfrak{M}_\alpha$ be an increasing net of von Neumann subalgebras of $\mathfrak{M}$ such that $\bigcup_\alpha\mathfrak{M}_\alpha$ generates $\mathfrak{M}$. Let ϕ and ψ be normal faithful positive linear functionals of $\mathfrak{M}$. Let ϕ_α and ψ_α be restrictions of ϕ and ψ to $\mathfrak{M}_\alpha$. Assume that*

(5.20) $$\psi\leqq k\phi$$

for some $0<k$. Then

(5.21) $$\lim_\alpha S(\phi_\alpha/\psi_\alpha)=S(\phi/\psi).$$

Proof. Let $\hat{\Phi}=\Phi\otimes e_{11}+\Psi\otimes e_{22}$ and $\hat{\phi}$ be as in (4.10) and (4.12). Let $\hat{\mathfrak{M}}=\mathfrak{M}\otimes\mathfrak{M}_2$, $\hat{\mathfrak{M}}_\alpha=\mathfrak{M}_\alpha\otimes\mathfrak{M}_2$, e_α be the projection on the closure of $\hat{\mathfrak{M}}_\alpha\hat{\Phi}$, $\hat{\Delta}$ be the modular operator for $\hat{\Phi}$ and $\hat{\Delta}_\alpha$ be the direct sum of the identity operator on $(\mathbb{1}-e_\alpha)(\mathfrak{H}\otimes\mathfrak{K})$ and the modular operator of $\hat{\Phi}$ for $\hat{\mathfrak{M}}_\alpha$ on $e_\alpha(\mathfrak{H}\otimes\mathfrak{K})$. By Theorem 2 of [2],

(5.22) $$\lim_\alpha(\mathbb{1}+\hat{\Delta}_\alpha)^{-1}=(\mathbb{1}+\hat{\Delta})^{-1}.$$

Hence

(5.23) $$\lim(u_{12}\hat{\Phi}, f_N(\log\hat{\Delta}_\alpha)u_{12}\hat{\Phi})=(u_{12}\hat{\Phi}, f_N(\log\hat{\Delta})u_{12}\hat{\Phi}),$$

where f_N is given by (2.11).

From

(5.24) $$\|\hat{\Delta}_\alpha^{1/2}u_{12}\hat{\Phi}\|^2=\|\hat{\Delta}^{1/2}u_{12}\hat{\Phi}\|^2=\|u_{12}^*\hat{\Phi}\|^2$$
$$=\phi(\mathbb{1}),$$

we obtain as in (2.12)

(5.25) $$0 \leqq \int_N^\infty (\log \lambda - \log N) \mathrm{d}(u_{12}\hat{\Phi}, E_\lambda^\alpha u_{12}\hat{\Phi}) \leqq \phi(\mathbb{1})(eN)^{-1},$$

(5.26) $$0 \leqq \int_N^\infty (\log \lambda - \log N) \mathrm{d}(u_{12}\hat{\Phi}, E_\lambda u_{12}\hat{\Phi}) \leqq \phi(\mathbb{1})(eN)^{-1}$$

for spectral projections E_λ^α and E_λ of $\hat{\Delta}_\alpha$ and $\hat{\Delta}$.

From $k\phi \geqq \psi$ and (4.23), we have

(5.27) $$\|\hat{\Delta}_\alpha^{-1/2} u_{12}\hat{\Phi}\|^2 = \psi(A_\alpha A_\alpha^*) \leqq k\psi(\mathbb{1}),$$

(5.28) $$\|\hat{\Delta}^{-1/2} u_{12}\hat{\Phi}\|^2 = \psi(AA^*) \leqq k\psi(\mathbb{1})$$

for some A_α and $A \in \mathfrak{M}$. Hence

(5.29) $$0 \geqq \int_0^{1/N} (\log \lambda + \log N) \mathrm{d}(u_{12}\hat{\Phi}, E_\lambda^\alpha u_{12}\hat{\Phi}) \geqq k\psi(\mathbb{1}) \inf_{\lambda \in [0, 1/N]} \lambda \log(N\lambda) \geqq -k\psi(\mathbb{1})(eN)^{-1},$$

(5.30) $$0 \geqq \int_0^{1/N} (\log \lambda + \log N) \mathrm{d}(u_{12}\hat{\Phi}, E_\lambda u_{12}\hat{\Phi}) \geqq -k\psi(\mathbb{1})/(Ne).$$

Collecting together (5.23), (5.25), (5.26), (5.29) and (5.30), we have

(5.31) $$\lim (u_{12}\hat{\Phi}, (\log \hat{\Delta}_\alpha) u_{12}\hat{\Phi}) = (u_{12}\hat{\Phi}, (\log \hat{\Delta}) u_{12}\hat{\Phi}).$$

Hence (5.21) holds due to

$$u_{12}\hat{\Phi} = \Psi \otimes e_{12}, \quad \hat{\Delta}_\alpha(f \otimes e_{12}) = (\Delta_{\Phi,\Psi} f) \otimes e_{12}$$

and independence of (1.1) on the choice of vector representatives.

Remark 1. Without the condition (5.20), we can obtain (5.23), (5.25) and (5.26). This implies

(5.32) $$\underline{\lim_{\alpha}}\, S(\phi_\alpha/\psi_\alpha) \geqq S(\phi/\psi).$$

If we have monotonicity, then (5.32) implies (5.21).

Remark 2. In the proof of Lemma 2 in [2], it is stated that

(5.33) $$\Delta_\alpha h_\alpha \Psi = 2h'\Psi - h_\alpha \Psi.$$

This is incorrect and should be corrected as follows:

The commutant of $\mathfrak{M}_\alpha$ on $\overline{\mathfrak{M}_\alpha \Psi}$ is $E_\alpha \mathfrak{M}_\alpha' E_\alpha$ where E_α is the projection on $\overline{\mathfrak{M}_\alpha \Psi}$ and belongs to $\mathfrak{M}_\alpha'$. Since $\phi \leqq \psi$ and ψ is faithful, there exists a unique $h_\alpha' \in E_\alpha \mathfrak{M}_\alpha' E_\alpha$ satisfying

(5.34) $$\phi(Q) = (h_\alpha' \Psi, Q\Psi), \qquad Q \in \mathfrak{M}_\alpha.$$

For this h_α' Lemma 1 of [2] is applicable and

(5.35) $$\Delta_\alpha h_\alpha \Psi = 2h_\alpha' \Psi - h_\alpha \Psi.$$

Since $E_\alpha Q\Psi = Q\Psi$ for $Q \in \mathfrak{M}_\alpha$ and $E_\alpha \Psi = \Psi$, (2.4) of [2] implies

(5.36) $$h_\alpha' = E_\alpha h' E_\alpha$$

satisfies (5.34). Hence

(5.37) $$\Delta_\alpha h_\alpha \Psi = 2E_\alpha h' \Psi - h_\alpha \Psi.$$

Since $E_\alpha \to \mathbf{1}$, we still have the conclusion of Lemma 2 in [2].

§6. Monotonicity for Case α

We start with lemmas which are needed in the proof.

Lemma 4. *Let $\mathfrak{A}$ be a von Neumann subalgebra of $\mathfrak{M}$ contained simultaneously in the centralizer of ϕ_1 and ϕ_2. Then*

(6.1) $$S(\phi_1/\phi_2) = S(E_{\mathfrak{N}}\phi_1/E_{\mathfrak{N}}\phi_2)$$

for $\mathfrak{N} = \mathfrak{A}' \cap \mathfrak{M}$.

Proof. Let $\hat{\Phi}$ and $\hat{\phi}$ be constructed as in (4.10) and (4.12). Let $\hat{\mathfrak{A}} = \mathfrak{A} \otimes \mathbf{1}$. Then $\hat{\mathfrak{A}}' \cap \hat{\mathfrak{M}} = \mathfrak{N} \otimes \mathfrak{M}_2$, by the commutant theorem. Since

$\mathfrak{A}$ is in the centralizer of ϕ_1 and ϕ_2, we have

(6.2) $$\hat{\phi}((x\otimes\mathbf{1})y)=\phi_1(xy_{11})+\phi_2(xy_{22})=\phi_1(y_{11}x)+\phi_2(y_{22}x)=\hat{\phi}(y(x\otimes\mathbf{1}))$$

for $x\in\mathfrak{A}$. Hence $\hat{\mathfrak{A}}$ is elementwise $\sigma_t^{\hat{\phi}}$ invariant. Hence $\hat{\mathfrak{N}}\equiv\mathfrak{N}\otimes\mathfrak{M}_2$ is $\sigma_t^{\hat{\phi}}$ invariant as a set. The state $\hat{\phi}$ restricted to $\hat{\mathfrak{N}}$ obviously satisfies the KMS condition relative to $\sigma_t^{\hat{\phi}}$ and hence $\sigma_t^{\hat{\phi}}$ coincides with the modular automorphisms of $\mathfrak{N}\otimes\mathfrak{M}_2$ for the state $\hat{\phi}$ restricted to $\hat{\mathfrak{N}}$. This also implies that $\overline{\mathfrak{N}\mathfrak{H}}\otimes\mathfrak{K}$ is $\Delta_{\hat{\Phi}}^{it}$ invariant and the restriction of $\Delta_{\hat{\Phi}}$ to this space is the modular operator $\Delta_{\hat{\Phi},\hat{\mathfrak{N}}}$ of $\hat{\Phi}$ for $\hat{\mathfrak{N}}$. Since $\mathbf{1}\otimes u_{12}\in\hat{\mathfrak{N}}$, we have

(6.3) $$\begin{aligned} S(E_{\mathfrak{N}}\phi_1/E_{\mathfrak{N}}\phi_2) &= -((\mathbf{1}\otimes u_{12})\hat{\Phi}, (\log\Delta_{\hat{\Phi},\hat{N}})(\mathbf{1}\otimes u_{12})\hat{\Phi}) \\ &= -((\mathbf{1}\otimes u_{12})\hat{\Phi}, (\log\Delta_{\hat{\Phi}})(\mathbf{1}\otimes u_{12})\hat{\Phi}) \\ &= S(\phi_1/\phi_2)\,. \end{aligned}$$

Lemma 5. *Let α_i be automorphisms of $\mathfrak{M}$. Let $\lambda_i\geqq 0$, $\Sigma\lambda_i=1$, and*

(6.4) $$\phi'=\Sigma\lambda_i\phi\circ\alpha_i, \qquad \psi'=\Sigma\lambda_i\psi\circ\alpha_i.$$

Then

(6.5) $$S(\phi'/\psi')\leqq S(\phi/\psi)\,.$$

Proof. The desired inequality (6.5) follows from the convexity of relative entropy if we prove

(6.6) $$S(\phi\circ\alpha/\psi\circ\alpha)=S(\phi/\psi)$$

for any automorphism α of $\mathfrak{M}$.

For any automorphism α of $\mathfrak{M}$, there exists by Theorem 11 of [3] a unitary U_α such that

(6.7) $$U_\alpha x U_\alpha^*=\alpha(x)\,,$$

(6.8) $$U_\alpha^*\xi(\chi)=\xi(\chi\circ\alpha)\,,$$

(6.9) $$[U_\alpha, J]=\mathbf{0},$$

where $\xi(\chi)$ is the unique vector representative of a normal positive linear functional χ on the fixed natural positive cone V.

From the definition

$$J\Delta^{1/2}_{\xi(\chi_1),\xi(\chi_2)}x\xi(\chi_2)=x^*\xi(\chi_1), \tag{6.10}$$

and the properties (6.7), (6.8), and (6.9), it follows that

$$U^*_\alpha\Delta^{1/2}_{\xi(\chi_1),\xi(\chi_2)}U_\alpha=\Delta^{1/2}_{\xi(\chi_1\circ\alpha),\xi(\chi_2\circ\alpha)}. \tag{6.11}$$

From (6.8) and (6.11), we obtain (6.6). Q.E.D.

We now prove the case α. Let $E_1\dots E_n$ be minimal projections of $\mathfrak{A}$ such that $\Sigma E_j=\mathbb{1}$. Let

$$\alpha_j(x)=(2E_j-\mathbb{1})x(2E_j-\mathbb{1}),\qquad x\in\mathfrak{M}, \tag{6.12}$$

which defines mutually commuting inner automorphisms α_j of $\mathfrak{M}$. Let

$$\phi'=2^{-n}\Sigma\phi\circ\alpha_1^{\sigma_1}\circ\cdots\circ\alpha_n^{\sigma_n}, \tag{6.13}$$

$$\psi'=2^{-n}\Sigma\psi\circ\alpha_1^{\sigma_1}\circ\cdots\circ\alpha_n^{\sigma_n}, \tag{6.14}$$

where the sum is over all possibilities for $\sigma_j=0$ or 1 and α_j^0 is an identity automorphism while $\alpha_j^1=\alpha_j$. The functionals ϕ' and ψ' are invariant under α_j for all j. Hence E_j are all in the centralizers of ϕ' and ψ'. By Lemmas 4 and 5, we have

$$S(E_{\mathfrak{N}}\phi/E_{\mathfrak{N}}\psi)=S(E_{\mathfrak{N}}\phi'/E_{\mathfrak{N}}\psi')$$
$$=S(\phi'/\psi')\leqq S(\phi/\psi).$$

This proves the monotonicity for the case $\mathfrak{N}=\mathfrak{A}'\cap\mathfrak{M}$ with a finite dimensional commutative subalgebra $\mathfrak{A}$.

§7. Monotonicity for Case β

We start from a special case and gradually go to a general case.

(1) *Commutative finite dimensional* $\mathfrak{N}_1$: Let $E_1\dots E_n$ be the minimal projections of $\mathfrak{N}_1$ such that $\Sigma E_i=\mathbb{1}$. Since $\mathfrak{N}_1$ is in the center of $\mathfrak{M}$, E_j are invariant under any modular automorphisms. Consequently, we

have

(7.1) $$\mathfrak{H} = \Sigma^{\oplus} E_j \mathfrak{H},$$

(7.2) $$\Phi_k = \Sigma^{\oplus} E_j \Phi_k, \qquad (k=1, 2),$$

(7.3) $$\Delta_{\Phi_1, \Phi_2} = \Sigma^{\oplus} \Delta_{E_j \Phi_1, E_j \Phi_2}.$$

Let

(7.4) $$\phi_{kj}(x) = \phi_k(E_j x), \qquad x \in \mathfrak{N}.$$

From (7.2) and (7.3), we have

(7.5) $$S(\phi_1/\phi_2) = \Sigma S(\phi_{1j}/\phi_{2j}).$$

We also have

(7.6) $$E_{\mathfrak{N}} \phi_k = \Sigma \phi_{kj} = \Sigma n^{-1}(n \phi_{kj}).$$

By convexity we have

(7.7) $$\begin{aligned} S(E_{\mathfrak{N}} \phi_1 / E_{\mathfrak{N}} \phi_2) &\leqq \Sigma n^{-1} S(n \phi_{1j} / n \phi_{2j}) \\ &= \Sigma S(\phi_{1j}/\phi_{2j}) \\ &= S(\phi_1/\phi_2), \end{aligned}$$

where we have used the homogeneity

(7.8) $$S(\lambda \phi / \lambda \psi) = \lambda S(\phi/\psi).$$

(2) *Commutative* $\mathfrak{N}_1$: Let $\mathfrak{A}_\alpha$ be the increasing net of all finite dimensional subalgebra of $\mathfrak{N}_1$. By Lemma 3, we have

(7.9) $$\begin{aligned} &\lim_\alpha S(E_{\mathfrak{N} \otimes \mathfrak{A}_\alpha}(\phi_1 + \varepsilon \phi_2) / E_{\mathfrak{N} \otimes \mathfrak{A}_\alpha}(\phi_2)) \\ &= S(\phi_1 + \varepsilon \phi_2 / \phi_2). \end{aligned}$$

By the previous case, we have

(7.10) $$\begin{aligned} &S(E_{\mathfrak{N}}(\phi_1 + \varepsilon \phi_2) / E_{\mathfrak{N}}(\phi_2)) \\ &\leqq S(E_{\mathfrak{N} \otimes \mathfrak{A}_\alpha}(\phi_1 + \varepsilon \phi_2) / E_{\mathfrak{N} \otimes \mathfrak{A}_\alpha}(\phi_2)) \end{aligned}$$

for all α and hence

$$S(E_{\mathfrak{N}}\phi_1+\varepsilon E_{\mathfrak{N}}\phi_2/E_{\mathfrak{N}}\phi_2) \leqq S(\phi_1+\varepsilon\phi_2/\phi_2). \tag{7.11}$$

By taking the limit $\varepsilon\to+0$, we obtain the monotonicity by Lemma 2.

(3) *Finite* $\mathfrak{N}_1$: Let

$$(p\psi)(y)=\psi(y^{\natural}), \qquad y\in\mathfrak{N}_1 \tag{7.12}$$

where ψ is a σ-weakly continuous linear functional on $\mathfrak{N}_1$ and $x^{\natural}$ denotes the unique conditional expectation from N_1 to its center $\mathfrak{Z}\equiv\mathfrak{N}_1\cap\mathfrak{N}_1'$ satisfying $(y_1y_2)^{\natural}=(y_2y_1)^{\natural}$. It is known ([9], Chapter 3, §5, Lemma 4 along with Radon Nikodym Theorem) that for any $\varepsilon>0$ and finite number of ψ_k, there exist inner automorphisms α_j of $\mathfrak{N}_1$ and $\lambda_j\geqq 0$ with $\Sigma\lambda_j=1$ satisfying $\|p\psi_k-\Sigma\lambda_j\psi_k\circ\alpha_j\|\leqq\varepsilon$ for all k.

The $\natural$-mapping extends to a normal expectation from $\mathfrak{N}\otimes\mathfrak{N}_1$ to $\mathfrak{N}\otimes\mathfrak{Z}$ satisfying $(x\otimes y)^{\natural}=x\otimes y^{\natural}$. Correspondingly p is defined for functionals on $\mathfrak{N}\otimes\mathfrak{N}_1$. Since products of normal linear functionals are total in norm topology, we also can approximate $p\phi_k$ by $\Sigma\lambda_j\phi_k\circ\alpha_j$ simultaneously for $k=1, 2$, where α_j is an inner automorphism by elements in $\mathfrak{N}_1$. By lower semi-continuity, convexity, and Lemma 5,

$$\begin{aligned} S(p\phi_1/p\phi_2) &\leqq \underline{\lim}\, S(\Sigma\lambda_j\phi_1\circ\alpha_j/\Sigma\lambda_j\phi_2\circ\alpha_j) \\ &\leqq \underline{\lim}\, \Sigma\lambda_j S(\phi_1\circ\alpha_j/\phi_2\circ\alpha_j) \\ &= S(\phi_1/\phi_2). \end{aligned} \tag{7.13}$$

By $(y_1y_2)^{\natural}=(y_2y_1)^{\natural}$, $\mathfrak{N}_1$ is in the centralizer of $p\phi_1$ and $p\phi_2$. Since $\mathfrak{N}_1'\cap\mathfrak{M}=\mathfrak{N}\otimes\mathfrak{Z}$, Lemma 4 implies

$$\begin{aligned} S(p\phi_1/p\phi_2) &= S(E_{\mathfrak{N}\otimes\mathfrak{Z}}(p\phi_1)/E_{\mathfrak{N}\otimes\mathfrak{Z}}(p\phi_2)) \\ &= S(E_{\mathfrak{N}\otimes\mathfrak{Z}}\phi_1/E_{\mathfrak{N}\otimes\mathfrak{Z}}\phi_2), \end{aligned} \tag{7.14}$$

where $\mathfrak{N}\otimes\mathfrak{Z}$ is elementwise invariant under $\natural$-mapping and hence $E_{\mathfrak{N}\otimes\mathfrak{Z}}(p\phi)=E_{\mathfrak{N}\otimes\mathfrak{Z}}\phi$. By combining (7.13), (7.14) and the previous case (2), we obtain the monotonicity for the present case.

(4) *General* $\mathfrak{N}_1$: Let ψ be a normal faithful positive linear functional on $\mathfrak{N}_1$ (for example restriction of ϕ_k to $\mathfrak{N}_1$). We first consider $\mathfrak{N}_1$ alone in a space $\mathfrak{H}_1$ with a cyclic and separating vector Ψ such that $(\Psi, y\Psi)=\psi(y)$, $y\in\mathfrak{N}_1$. Let E_0 be the projection onto the subspace of all Δ_Ψ invariant vectors.

$$\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\Delta_\Psi^{it}\mathrm{d}t=E_0 \tag{7.15}$$

strongly. Hence

$$p_\psi(y)\equiv\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}\sigma_t^\psi(y)\,\mathrm{d}t\in\mathfrak{M} \tag{7.16}$$

is strongly convergent on Ψ and hence on $\mathfrak{M}'\Psi$ and hence on $\mathfrak{H}_1$ by the uniform boundedness. The mapping p_ψ is the conditional expectation from $\mathfrak{N}_1$ to the centralizer $\mathfrak{N}_2=\mathfrak{N}_1^\psi$ relative to ψ.

If $\phi\leqq\lambda\psi$ for some $\lambda>0$, then there exists $y'\in\mathfrak{N}_1'$ such that

$$\phi(y)=(\Psi, y\,y'\Psi).$$

Then

$$\frac{1}{2T}\int_{-T}^{T}(\phi\circ\sigma_t^\psi)(y)\,\mathrm{d}t=(\Psi, y(2T)^{-1}\int_{-T}^{T}\Delta_\Psi^{it}y'\Psi) \tag{7.17}$$

converges in norm of linear functionals simultaneously for a finite number of such ϕ's. Since ϕ satisfying $\phi\leqq\lambda\psi$ for some λ is norm dense, it is possible to approximate $\phi_k\circ p_\psi$ simultaneously for a finite number of ϕ_k by $\Sigma\lambda_i\phi_k\circ\sigma_{t_i}^\psi$ where $\lambda_i\geqq 0$, $\Sigma\lambda_i=1$. Hence the same holds for functionals ϕ_k on $\mathfrak{N}\otimes\mathfrak{N}_1$ where we approximate $\phi_k\circ(\iota\otimes p_\psi)$ by $\Sigma\lambda_i\phi_k\circ(\iota\otimes\sigma_{t_i}^\psi)$ with ι denoting the identity automorphism. Hence

$$S(\phi_1\circ(\iota\otimes p_\psi)/\phi_2\circ(\iota\otimes p_\psi)) \leqq S(\phi_1/\phi_2). \tag{7.18}$$

On the other hand, $\mathfrak{N}_2=\mathfrak{N}_1^\psi$ is a finite algebra with ψ as a trace. By the previous case (3), we have

$$S(E_\mathfrak{N}\phi_1/E_\mathfrak{N}\phi_2)\leqq S(E_{\mathfrak{N}\otimes\mathfrak{N}_2}\{\phi_1\circ(\iota\otimes p_\psi)\}/E_{\mathfrak{N}\otimes\mathfrak{N}_2}\{\phi_2\circ(\iota\otimes p_\psi)\}) \tag{7.19}$$

where we have used $E_{\mathfrak{N}}\phi_k = E_{\mathfrak{N}}\{\phi_k \circ (\iota \otimes p_\psi)\}$.

We can complete our proof if we show

$$(7.20) \qquad S(E_{\mathfrak{N}\otimes\mathfrak{N}_2}\{\phi_1 \circ (\iota\otimes p_\psi)\}/E_{\mathfrak{N}\otimes\mathfrak{N}_2}\{\phi_2 \circ (\iota\otimes p_\psi)\}) = S(\phi_1 \circ (\iota\otimes p_\psi)/\phi_2 \circ \iota\otimes p_\psi).$$

Noting that $\mathfrak{N}\otimes\mathfrak{N}_2$ is the set of σ_t^ψ-invariant elements in $\mathfrak{N}\otimes\mathfrak{N}_1$ and that both $\phi_k \circ (\iota\otimes p_\psi)$ are invariant under $\iota\otimes\sigma_t^\psi$, $t\in\mathbb{R}$, (7.20) follows from the following:

Lemma 6. *Let $\mathscr{G}$ be a set of automorphisms of $\mathfrak{M}$ such that ϕ_1 and ϕ_2 are both $\mathscr{G}$-invariant, i.e. $\phi_k \circ g = \varphi_k$ for all $g\in\mathscr{G}$. Let $\mathfrak{N} = \mathfrak{M}^G$ be the set of $\mathscr{G}$-invariant elements of $\mathfrak{M}$. Then*

$$(7.21) \qquad S(E_{\mathfrak{N}}\phi_1/E_{\mathfrak{N}}\phi_2) = S(\phi_1/\phi_2).$$

Proof. Let $\hat{\Phi}$ and $\hat{\phi}$ be given by (4.10) and (4.12). Then $\hat{\phi}$ is invariant under automorphisms $g\otimes\iota$ on $\mathfrak{M}\otimes\mathfrak{M}_2$ for all $g\in\mathscr{G}$. Hence $g\otimes\iota$ commutes with $\sigma_t^{\hat{\phi}}$. This implies that $\mathfrak{N}\otimes\mathfrak{M}_2$ which is the set of $(\mathscr{G}\otimes\iota)$-invariant elements of $\mathfrak{M}\otimes\mathfrak{M}_2$ is $\sigma_t^{\hat{\phi}}$ invariant as a set. By the same proof as Lemma 4, we obtain (7.21).

§8. Monotonicity for Case γ

First we consider finite dimensional $\mathfrak{N}$. Let $E_1,\ldots,E_n$ be the minimal projections of the center of $\mathfrak{N}$ satisfying $\Sigma E_j = \mathbb{1}$. Since $\mathfrak{A} = \{E_1,\ldots, E_n\}''$ is commutative, we have

$$(8.1) \qquad S(E_{\mathfrak{A}_1}\phi_1/E_{\mathfrak{A}_1}\phi_2) \leqq S(\phi_1/\phi_2)$$

for $\mathfrak{A}_1 = \mathfrak{A}'\cap\mathfrak{M}$.

The algebra $\mathfrak{A}_1$ is a direct sum of $\mathfrak{A}_1 E_j$ and each $\mathfrak{A}_1 E_j$ is a tensor product $(\mathfrak{N}E_j)\otimes\{(\mathfrak{N}'\cap\mathfrak{A}_1)E_j\}$. Let ϕ_{kj} be the restriction of ϕ_k to $\mathfrak{A}_1 E_j$, where E_j is the identity. As in (7.5), we have

$$(8.2) \qquad S(\phi_1/\phi_2) = \sum_j S(\phi_{1j}/\phi_{2j}).$$

$$(8.3) \qquad S(E_{\mathfrak{N}}\phi_1/E_{\mathfrak{N}}\phi_2) = \sum_j S(E_{\mathfrak{N}E_j}(\phi_{1j})/E_{\mathfrak{N}E_j}(\phi_{2j})).$$

By case (β), we have

$$S(\phi_{1j}/\phi_{2j}) \geqq S(E_{\mathfrak{N}E_j}(\phi_{1j})/E_{\mathfrak{N}E_j}(\phi_{2j})). \tag{8.4}$$

From (8.2), (8.3) and (8.4), we have monotonicity.

The case of general approximately finite algebra $\mathfrak{N}$ can be deduced from the case of finite dimensional $\mathfrak{N}$ by using Lemmas 2 and 3 just as in the proof of case $\beta(2)$.

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