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Relative Entropy for States of von Neumann Algebras II

By

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Abstract

Earlier definition of the relative entropy of two faithful normal positive linear functionals of a von Neumann algebra is generalized to non-faithful functionals. Basic properties of the relative entropy are proved for this generalization.

§ 1. Introduction

For two faithful normal positive linear functionals ϕ and ψ of a von Neumann algebra M, the relative entropy $S(\psi|\phi)$ is defined and its properties are proved in an earlier paper [1].

When M is a finite dimensional factor, it is given by

$$S(\psi|\phi) = \phi(\log\rho_\phi - \log\rho_\psi) \tag{1.1}$$

where ρ_ϕ and ρ_ψ are density matrices for ϕ and ψ. If ψ and ϕ are faithful, ρ_ϕ and ρ_ψ are strictly positive and (1. 1) clearly makes sense. However the first term of (1. 1) always makes sense (under the convention $\lambda\log\lambda=0$ for $\lambda=0$) and the second term is either finite or infinite. Therefore (1. 1) can be given an unambiguous finite or positive infinite value for every ϕ and ψ.

We shall make corresponding generalization for an arbitrary von Neumann algebra M and any normal positive linear functionals ψ and ϕ. We shall also define the relative entropy of two positive linear functionals of a C^*-algebra $\mathfrak{A}$ and give an alternative proof of a result of [2]. For the latter case, we relate the conditional entropy introduced in [3] with our relative entropy.

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The relative entropy for non-faithful functionals will be shown to satisfy all properties proved for faithful functionals in [1]. Some of these properties will be applied to a discussion of local thermodynamical stability in [3].

For simplicity, we shall assume that M has a faithful normal state although many of the results are independent of this assumption.

§ 2. Relative Modular Operator

Let Φ and Ψ be vectors in a natural positive cone V([4], [5], [6]) for a von Neumann algebra M on a Hilbert space H and let ϕ and ψ be the corresponding normal positive linear functionals of M. Let $s^R(\Omega)$ denote the R-support of a vector Ω, where R is a von Neumann algebra.

Definition 2. 1. Operators $S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ with their domains

$$D(S_{\Phi,\Psi})=M\Psi+(\mathbf{1}-s^{M'}(\Psi))H,$$

$$D(F_{\Phi,\Psi})=M'\Psi+(\mathbf{1}-s^{M}(\Psi))H,$$

are defined by

$$S_{\Phi,\Psi}\{x\Psi+\Omega\}=s^M(\Psi)x^*\Phi, \tag{2.1}$$

$$F_{\Phi,\Psi}\{x'\Psi+\Omega'\}=s^{M'}(\Psi)x'^*\Phi, \tag{2.2}$$

where $x\in M$, $x'\in M'$, $s^{M'}(\Psi)\Omega=0$, $s^M(\Psi)\Omega'=0$.

Lemma 2. 2. *$S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ are closable antilinear operators.*

Proof: If $x_1\Psi+\Omega_1=x_2\Psi+\Omega_2$ for $x_1, x_2\in M$ and $\Omega_1, \Omega_2\in(\mathbf{1}-s^{M'}(\Psi))H$, then $\Omega_1=\Omega_2$ and $(x_1-x_2)s^M(\Psi)=\mathbf{0}$, so that $s^M(\Psi)x_1{}^*\Phi=s^M(\Psi)x_2{}^*\Phi$. This shows that $S_{\Phi,\Psi}$ is well-defined. Then it is clearly antilinear. Similarly $F_{\Phi,\Psi}$ is an antilinear operator.

Let $x\in M$, $x'\in M'$, $s^{M'}(\Psi)\Omega=s^M(\Psi)\Omega'=0$. Then

$$(S_{\Phi,\Psi}\{x\Psi+\Omega\}, \{x'\Psi+\Omega'\})=(x^*\Phi, x'\Psi)$$

$$=(F_{\Phi,\Psi}\{x'\Psi+\Omega\}, \{x\Psi+\Omega\}).$$

Since $S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$ have dense domains, this shows the closability of

$S_{\Phi,\Psi}$ and $F_{\Phi,\Psi}$.

Definition 2.3. The *relative modular operator* $\Delta_{\Phi,\Psi}$ is defined by

(2.3) $$\Delta_{\Phi,\Psi} = (S_{\Phi,\Psi})^* \bar{S}_{\Phi,\Psi}$$

where the bar denotes the closure.

We denote by J the modular conjugation operator associated with the natural positive cone V.

Theorem 2.4.

(1) *The kernel of* $\Delta_{\Phi,\Psi}$ *is* $\mathbf{1} - s^{M'}(\Psi) s^M(\Phi)$.

(2) *The following formulas hold, where the bar denotes the closure.*

(2.4) $$\bar{S}_{\Phi,\Psi} = J(\Delta_{\Phi,\Psi})^{1/2}, \quad \bar{F}_{\Phi,\Psi} = (\Delta_{\Phi,\Psi})^{1/2} J,$$

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(2.5) $$J\Delta_{\Psi,\Phi} J\Delta_{\Phi,\Psi} = \Delta_{\Phi,\Psi} J\Delta_{\Psi,\Phi} J = s^{M'}(\Psi) s^M(\Phi).$$

(3) *If* $s^M(\Phi_1) \perp s^M(\Phi_2)$, then

(2.6) $$\Delta_{\Phi_1+\Phi_2,\Psi} = \Delta_{\Phi_1,\Psi} + \Delta_{\Phi_2,\Psi}.$$

Proof:

(1) and (2): First we prove Theorem for the special case $\Phi = \Psi$. The domain of $S_{\Psi,\Psi}$ is split into a direct sum of 3 parts:

$$D(S_{\Psi,\Psi}) = s^M(\Psi) M\Psi + (\mathbf{1} - s^M(\Psi)) M\Psi + (\mathbf{1} - s^{M'}(\Psi)) H.$$

Accordingly, we split $S_{\Psi,\Psi}$ as a direct sum

$$S_{\Psi,\Psi} = \hat{S}_{\Psi,\Psi} \oplus \mathbf{0} \oplus \mathbf{0}$$

where $S_{\Psi,\Psi}$ is the operator on $s^M(\Psi)\ s^{M'}(\Psi) H$ defined by

$$\hat{S}_{\Psi,\Psi} x\Psi = x^*\Psi, \quad x \in s^M(\Psi) M s^M(\Psi)$$

and the splitting of the Hilbert space is

$$H = s^M(\Psi) s^{M'}(\Psi) H \oplus (\mathbf{1} - s^M(\Psi)) s^{M'}(\Psi) H \oplus (\mathbf{1} - s^{M'}(\Psi)) H.$$

Since Ψ is cyclic and separating relative to $s^M(\Psi) M s^M(\Psi)$ in the subspace $s^M(\Psi) s^{M'}(\Psi) H$,

(2.7) $$\Delta_{\Psi,\Psi} = \tilde{\Delta}_{\Psi,\Psi} \oplus \mathbf{0} \oplus \mathbf{0}$$

where $\tilde{\Delta}_{\Psi,\Psi}$ is the modular operator of Ψ relative to $s^M(\Psi)Ms^M(\Psi)$. Since[1] $s^{M'}(\Psi)=J\,s^M(\Psi)J$, J commutes with $s^M(\Psi)\ s^{M'}(\Psi)$ and hence leaves $s^M(\Psi)s^{M'}(\Psi)H$ invariant. The restriction of J to this subspace is the modular conjugation operator for Ψ, as can be checked by the characterization of J given in [4]. Therefore the known property of the modular operator for a cyclic and separating vector implies (1) and (2) for the case $\Psi=\Phi$.

To prove (1) and (2) for the general case, we use the 2×2 matrix method of Connes [7]. Let $\tilde{M}=M\otimes M_2$ with M_2 a type I_2 factor on a 4-dimensional space K, let u_{ij} be a matrix unit of M_2, let e_{ij} be an orthonormal basis of K satisfying $u_{ij}e_{kl}=\delta_{jk}e_{il}$, let J_K be the modular conjugation operator of $e_{11}+e_{22}$ (i.e. $J_Ke_{ij}=e_{ji}$), and let

$$\Omega=\sum \Omega_j\otimes e_{jj} \tag{2.8}$$

with $\Omega_1=\Psi$ and $\Omega_2=\Phi$. From definition, we obtain

$$\begin{aligned} s^{\tilde{M}}(\Omega) &= \sum s^M(\Omega_j)\otimes u_{jj}\,, \\ s^{\tilde{M}'}(\Omega) &= \sum s^{M'}(\Omega_j)\otimes J_Ku_{jj}J_K\,, \\ (\mathbf{1}\otimes u_{ii})S_{\Omega,\Omega}(\mathbf{1}\otimes u_{jj}) &= S_{\Omega_j,\Omega_i}\otimes u_{ii}J_Ku_{jj}\,. \end{aligned} \tag{2.9}$$

Since the modular conjugation operator J for the natural positive cone of M containing $V\otimes(e_1+e_2)$ is given by $J\otimes J_K$, we obtain

$$\Delta_{\Omega,\Omega}(\mathbf{1}\otimes J_Ku_{ii}J_Ku_{jj})=\Delta_{\Omega_j,\Omega_i}\otimes J_Ku_{ii}J_Ku_{jj}\,. \tag{2.10}$$

Hence (1) and (2) proved above for $\Delta_{\Omega,\Omega}$ imply the same for $\Delta_{\Phi,\Psi}$ and $\Delta_{\Psi,\Phi}$.

(3) If $s^M(\Phi_j)$ is mutually orthogonal for $j=1,2$, then the same holds for $s^{M'}(\Phi_j)=Js^M(\Phi_j)J$. By (1) and (2), the range projection of $S_{\Phi_j\Psi}$ is

$$Js^{M'}(\Psi)s^M(\Phi_j)J=s^M(\Psi)s^{M'}(\Phi_j)$$

and is mutually orthogonal for $j=1,2$. The same holds for the corange projection. From definition we obtain

$$S_{\Phi_1+\Phi_2,\Psi}=S_{\Phi_1,\Psi}+S_{\Phi_2,\Psi}\,. \tag{2.11}$$

Hence we obtain (2.6). Q.E.D.

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[1] This follows from $J\Psi=\Psi$.

§ 3. Relative Entropy for States of von Neumann Algebras

Let M, Ψ, Φ, ψ and ϕ be as in the previous section. Let $E_\lambda^{\Phi,\Psi}$ denote the spectral projections of $\Delta_{\Phi,\Psi}$, $s(\omega)$ denote the support of the positive linear functional ω.

Definition 3.1. For $\phi \neq 0$, the relative entropy $S(\psi/\phi)$ is defined by

$$S(\psi/\phi)\begin{cases} = \int_{+0}^{\infty} \log \lambda d(\Phi, E_\lambda^{\Phi,\Psi}\Phi) & \text{if } s(\psi) \geqq s(\phi). \\ = +\infty & \text{otherwise}. \end{cases}$$

Lemma 3.2. $S(\psi/\phi)$ is well defined, takes finite value or $+\infty$ and satisfies

$$S(\psi/\phi) \geqq -\phi(\mathbf{1}) \log\{\psi(s(\phi))/\phi(\mathbf{1})\}. \tag{3.1}$$

Proof: First consider the case $s(\psi) \geqq s(\phi)$. Since $s^M(\Psi) = s(\psi) \geqq s(\phi) = s^M(\Phi)$, we have $S_{\Phi,\Psi}\Psi = \Phi$.

Since $J\Phi = \Phi$, we have $(\Delta_{\Phi,\Psi})^{1/2}\Psi = \Phi$. Hence

$$\begin{aligned} \int_{+0}^{\infty} \lambda^{-1} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi) &= (\Psi, (\mathbf{1} - E_{+0}^{\Phi,\Psi})\Psi) \\ &= (\Psi, s^{M'}(\Psi) s^M(\Phi)\Psi) \\ &= (\Psi, s(\phi)\Psi) = \psi(s(\phi)) \leqq 1. \end{aligned} \tag{3.2}$$

This implies that the integral defining $S(\psi/\phi)$ converges at the lower end. Hence it is well defined and takes either finite value or $+\infty$.

Since $s(\psi) \geqq s(\phi)$ implies

$$\int_{+0}^{\infty} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi) = (\Phi, s^{M'}(\Psi) s^M(\Phi)\Phi) = \phi(\mathbf{1}),$$

$\mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1})$ is a probability measure on $(0, +\infty)$. By the concavity of the logarithm, we obtain

$$\begin{aligned} S(\psi/\phi) &= -\phi(\mathbf{1}) \int_{+0}^{\infty} \log(\lambda^{-1}) \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1}) \\ &\geqq -\phi(\mathbf{1}) \log\left\{\int_{+0}^{\infty} \lambda^{-1} \mathrm{d}(\Phi, E_\lambda^{\Phi,\Psi}\Phi)/\phi(\mathbf{1})\right\} \end{aligned}$$

$$= -\phi(\mathbf{1})\log\{\psi(s(\phi))/\phi(\mathbf{1})\}.$$

The statement of Lemma holds trivially for the case where $s(\psi)\geqq s(\phi)$ does not hold.

Remark 3.3. The definition of $S(\psi/\phi)$ uses the (unique) vector representatives Ψ and Φ in a natural positive cone V. The value $S(\psi/\phi)$, however does not depend on the choice of the natural positive cone V because of the following reason. If V′ is another natural positive cone, then there exists a unitary $w'\in M'$ such that $V'=w'V$. $\Psi'=w'\Psi$ and $\Phi'=w'\Phi$ are representative vectors of ψ and ϕ in V'. We then obtain $\Delta_{\Phi',\Psi'}=w'\Delta_{\Phi,\Psi}(w')^*$ and hence $S(\psi/\phi)$ is unchanged.

Remark 3.4. By Theorem 2.4 (2), we have

$$\{\log\Delta_{\Phi,\Psi}+J(\log\Delta_{\Psi,\Phi})J\}s^{M}(\Phi)s^{M'}(\Psi)=\mathbf{0}.$$

Hence, for the case $s(\psi)\geqq s(\phi)$, we obtain the following expression ([1]):

$$S(\psi/\phi)=-(\Phi,\log\Delta_{\Psi,\Phi}\Phi). \tag{3.3}$$

Remark 3.5. If $s(\psi)=s(\phi)$, then $\Delta_{\Phi,\Psi}$ is $\mathbf{0}$ on $(\mathbf{1}-s(\psi)Js(\psi)J)H$ and coincides with the relative modular operator for $s(\psi)Ms(\psi)$ on the space $s(\psi)Js(\psi)JH$, where Φ and Ψ are cyclic and separating for $s(\psi)Ms(\psi)$. Hence $S(\psi/\phi)$ in this case is the same as the relative entropy of two *faithful* normal positive linear functionals ψ and ϕ of $s(\psi)Ms(\psi)$.

Theorem 3.6.

(1) *If* $\psi(\mathbf{1})=\phi(\mathbf{1})>0$, *then* $S(\psi/\phi)\geqq 0$. *The equality* $S(\psi/\phi)=0$ *holds if and only if* $\psi=\phi$.

(2) *If* $s(\phi_1)\perp s(\phi_2)$, *then*

$$S(\psi/\phi_1)+S(\psi/\phi_2)=S(\psi/\phi_1+\phi_2). \tag{3.4}$$

(3) *For* $\lambda_1,\lambda_2>0$,

$$S(\lambda_1\psi/\lambda_2\phi)=\lambda_2 S(\psi/\phi)-\lambda_2\phi(\mathbf{1})\log(\lambda_1/\lambda_2). \tag{3.5}$$

(4) *If* $\psi_1\geqq\psi_2$, *then*

(3. 6) $$S(\psi_1/\phi) \leqq S(\psi_2/\phi).$$

Proof.

(1) Since $\psi(s(\phi)) \leqq \psi(\mathbf{1})$, the assumption $\psi(\mathbf{1}) = \phi(\mathbf{1})$ and (3. 1) imply $S(\psi/\phi) \geqq 0$. Furthermore, the equality $S(\psi/\phi) = 0$ holds only if $s(\psi) \geqq s(\phi)$ and $\psi(s(\phi)) = \psi(\mathbf{1})$. We then have $s(\phi) = s(\psi)$; hence Remark 3. 5 and the strict positivity of $S(\psi/\phi)$ for faithful ψ and ϕ ([1]) imply $\phi = \psi$ also in the present case. Conversely $\phi = \psi$ implies $S(\psi/\phi) = 0$.

(2) (3. 4) follows from (2. 6) and Definition 3. 1.

(3) The vector representatives for $\lambda_1\psi$ and $\lambda_2\phi$ differs from those for ψ and ϕ by factors $(\lambda_1)^{1/2}$ and $(\lambda_2)^{1/2}$ respectively. Hence this induces a change of $S_{\Phi,\Psi}$ by a factor $(\lambda_2/\lambda_1)^{1/2}$ and a change of $\Delta_{\Phi,\Psi}$ by a factor (λ_2/λ_1). The latter proves (3. 5).

(4) If $s(\psi_2) \geqq s(\phi)$ does not hold, then (3. 6) is trivially true. Hence we assume $s(\psi_2) \geqq s(\phi)$. Since $\psi_1 \geqq \psi_2$ implies $s(\psi_1) \geqq s(\psi_2)$, we also have $s(\psi_1) \geqq s(\phi)$. The following proof is then the same as that for the case of faithful ψ's and ϕ:

Denoting representative vectors of ψ_1, ψ_2 and ϕ in the natural positive cone by Ψ_1, Ψ_2 and Φ, respectively, we obtain

$$\begin{aligned}\|(\Delta_{\Psi_1,\Phi})^{1/2}x\Phi\|^2 &= \|S_{\psi_1,\Phi}x\Phi\|^2 = \|s(\phi)x^*\Psi_1\|^2\\ &= \psi_1(xs(\phi)x^*) \geqq \psi_2(xs(\phi)x^*) = \|(\Delta_{\Psi_2,\Phi})^{1/2}x\Phi\|^2,\end{aligned}$$

for all $x \in M$. Since both $(\Delta_{\Psi_j,\Phi})^{1/2}$ vanish on $(s^{M'}(\Phi)H)^\perp$ and since $M\Phi + (\mathbf{1} - s^{M'}(\Phi))H$ is the core of $(\Delta_{\Psi_1,\Phi})^{1/2}$, it follows that the domain of $(\Delta_{\Psi_1,\Phi})^{1/2}$ is contained in the domain of $(\Delta_{\Psi_2,\Phi})^{1/2}$ and for all Ω in the domain of $(\Delta_{\Psi_1,\Phi})^{1/2}$

$$\|(\Delta_{\Psi_1,\Phi})^{1/2}\Omega\| \geqq \|(\Delta_{\Psi_2,\Phi})^{1/2}\Omega\|.$$

Hence

$$\|(\Delta_{\Psi_1,\Phi} + r)^{1/2}\Omega\|^2 \geqq \|(\Delta_{\Psi_2,\Phi} + r)^{1/2}\Omega\|^2$$

for all such Ω and $r > 0$. Taking $\Omega = (\Delta_{\Psi_1,\Phi} + r)^{-1/2}\Omega'$ with an arbitrary Ω', we find

$$\|(\Delta_{\Psi_2,\Phi}+r)^{1/2}(\Delta_{\Psi_1,\Phi}+r)^{-1/2}\|\leqq 1\,.$$

Taking adjoint operator acting on $\Omega=(\Delta_{\Psi_2,\Phi}+r)^{-1/2}\Omega'$ with an arbitrary Ω', we find

$$\|(\Delta_{\Psi_1,\Phi}+r)^{-1/2}\Omega'\|^2\leqq\|(\Delta_{\Psi_2,\Phi}+r)^{-1/2}\Omega'\|^2$$

and hence

$$(3.7)\qquad (\Delta_{\Psi_1,\Phi}+r)^{-1}\leqq(\Delta_{\Psi_2,\Phi}+r)^{-1}\,.$$

By (3.3) we have

$$(3.8)\quad S(\psi_j/\phi)=-\int_0^\infty\left\{\int_0^\infty[(1+r)^{-1}-(\lambda+r)^{-1}]\mathrm{d}r\right\}\mathrm{d}(\Phi, E_\lambda^{\Psi_j,\Phi}\Phi)$$
$$=\int_0^\infty(\Phi,[(r+\Delta_{\Psi_j,\Phi})^{-1}-(1+r)^{-1}]\Phi)\,dr$$

where $E_\lambda^{\Psi_j,\Phi}$ is the spectral projection of $\Delta_{\Psi_j,\Phi}$ and the interchange of r- and λ- integrations are allowed because the double integral is definite in the Lebesque sense (finite or $+\infty$) due to

$$\int_0^\infty\lambda\mathrm{d}(\Phi, E_\lambda^{\Psi_j,\Phi}\Phi)=\|(\Delta_{\Psi_j,\Phi})^{1/2}\Phi\|^2=\|s(\phi)\Psi_j\|^2<\infty\,.$$

The equations (3.8) and (3.7) imply (3.6). Q.E.D.

The following Theorem describes the continuity property of $S(\psi/\phi)$ as a function of ψ and ϕ. (It is the same as the case of faithful ψ and ϕ.)

Theorem 3.7.

Assume that $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$.

(1) $\liminf S(\psi_\alpha/\phi_\alpha)\geqq S(\psi/\phi)$ (*the lower semicontinuity*).

(2) *If* $\lambda\psi_\alpha\geqq\phi_\alpha$ *for a fixed* $\lambda>0$, *then*

$$\lim S(\psi_\alpha/\phi_\alpha)=S(\psi/\phi)\,.$$

(3) *If* ψ_α *is monotone decreasing, then*

$$\lim S(\psi_\alpha/\phi)=S(\psi/\phi)\,.$$

We shall give proof of this Theorem in the next section. Using this theorem in an approximation argument, we obtain the next theorem from the same theorem ([1]) for faithful functionals.

Theorem 3.8.

(1) *$S(\psi/\phi)$ is jointly convex in ψ and ϕ.*

(2) *Let N be a von Neumann subalgebra of M and $E_N\omega$ denotes the restriction of a functional ω to N. Then*

$$S(E_N\psi/E_N\phi)\leqq S(\psi/\phi) \tag{3.9}$$

if N is any one of the following type:

(α) *$N=\mathfrak{A}'\cap M$ for a finite dimensional abelian $*$-subalgebra $\mathfrak{A}$ of M.*

(β) *$M=N\otimes N_1$.*

(γ) *N is approximately finite.*

Proof.

(1) We have to prove the following

$$S\Big(\sum_{j=1}^{n}\lambda_j\psi_j/\sum_{j=1}^{n}\lambda_j\phi_j\Big)\leqq\sum_{j=1}^{n}\lambda_jS(\psi_j/\phi_j) \tag{3.10}$$

for $\lambda_j>0$, $\sum\lambda_j=1$. Let $\psi=\sum\lambda_j\psi_j$, $\phi=\sum\lambda_j\phi_j$, $\omega=\psi+\phi$. By Remark 3.5,

$$S(\psi+\varepsilon\omega/\phi+\eta\omega)\leqq\sum_{j=1}^{n}\lambda_jS(\psi_j+\varepsilon\omega/\phi_j+\eta\omega)$$

follows from the convexity of $S(\psi_0/\phi_0)$ for faithful ψ_0 and ϕ_0. We first take the limit $\eta\to+0$ using Theorem 3.7 (2) and then take the limit $\varepsilon\to+0$ using Theorem 3.7 (3) to obtain (3.10).

(2) Let ω_0 be a faithful normal state of M and let $\omega=\omega_0+\phi+\psi$. Then

$$S(E_N(\psi+\varepsilon\omega)/E_N(\phi+\eta\omega))\leqq S(\psi+\varepsilon\omega/\phi+\eta\omega).$$

Again Theorem 3.7 (2) and (3) yield (3.9). Q.E.D.

The following Theorem describe some continuity property of $S(E_N\psi/E_N\phi)$ on N.

Theorem 3.9. *Let N_α be monotone increasing net of von Neumann subalgebras of M generating M.*

(1) $\liminf S(E_{N_\alpha}\psi/E_{N_\alpha}\phi)\geqq S(\psi/\phi)$.

(2) *If N_α is an AF algebra for all α, then*

$$\lim S(E_{N_\alpha}\psi/E_{N_\alpha}\phi)=S(\psi/\phi).$$

Proof of (1) and (2) will be given in the next section. (2) follows from (1) and Theorem 3.8 (2) (γ).

Let ψ be a faithful normal positive linear functional of M corresponding to a cyclic and separating vector Ψ and $h=h^*\in M$. Let $\Psi(h)$ denote the perturbed vector defined by (4.1) in [8]. Let ψ^h denote the perturbed state defined by

$$\psi^h(x)=(\Psi(h), x\Psi(h)),\quad x\in M.$$

Theorem 3.10.

$$S(\psi^h/\phi)=-\phi(h)+S(\psi/\phi),$$

$$S(\phi/\psi^h)=\psi^h(h)+S(\phi^h/\psi^h).$$

§ 4. Some Continuity Properties

We first prove some continuity properties of the relative modular operators.

Lemma 4.1. *If* $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$, *then*

$$\lim(r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2})^{-1}s^{M'}(\Psi)=(r+(\Delta_{\Phi,\Psi})^{1/2})^{-1}s^{M'}(\Psi) \tag{4.1}$$

for $r>0$ *and the convergence is uniform in* r *if* r *is restricted to any compact subset of* $(0,\infty)$, *where* Φ_α, Ψ_α, Φ *and* Ψ *are the representative vectors of* ϕ_α, ψ_α, ϕ *and* ψ *in the positive natural cone, respectively.*

Proof. The condition $\lim\|\phi_\alpha-\phi\|=\lim\|\psi_\alpha-\psi\|=0$ implies (Theorem 4(8) in [4])

$$\lim\|\Phi_\alpha-\Phi\|=\lim\|\Psi_\alpha-\Psi\|=0. \tag{4.2}$$

For $x'\in M'$, we have

$$\begin{aligned}&\lim\|s^M(\Psi_\alpha)x'\Psi-x'\Psi\|\\&\qquad=\lim\|s^M(\Psi_\alpha)x'(\Psi-\Psi_\alpha)+x'(\Psi_\alpha-\Psi)\|=0.\end{aligned}$$

Hence

$$\lim s^M(\Psi_\alpha) s^M(\Psi) = s^M(\Psi). \tag{4.3}$$

For $x \in M\ s^M(\Psi)$, we have

$$\begin{aligned}
&\| (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2} x\Psi_\alpha - (\Delta_{\Phi, \Psi})^{1/2} x\Psi \| \\
&\quad = \| J(\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2} x\Psi_\alpha - J(\Delta_{\Phi, \Psi})^{1/2} x\Psi \| \\
&\quad = \| s^M(\Psi_\alpha) x^* \Phi_\alpha - x^* \Phi \| \\
&\quad \leqq \| (s^M(\Psi_\alpha) s^M(\Psi) - s^M(\Psi)) x^* \Phi \| + \| s^M(\Psi_\alpha) x^* (\Phi_\alpha - \Phi) \| \to 0 .
\end{aligned}$$

Hence

$$\begin{aligned}
&\| \{ [\mathbb{1} + (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2}]^{-1} - [\mathbb{1} + (\Delta_{\Phi, \Psi})^{1/2}]^{-1} \} [\mathbb{1} + (\Delta_{\Phi, \Psi})^{1/2}] x\Psi \| \\
&\quad = \| [\mathbb{1} + (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2}]^{-1} \\
&\qquad \times \{ x(\Psi - \Psi_\alpha) + [(\Delta_{\Phi, \Psi})^{1/2} x\Psi - (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2} x\Psi_\alpha] \} + x(\Psi_\alpha - \Psi) \| \\
&\quad \leqq 2 \| x(\Psi_\alpha - \Psi) \| + \| (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2} x\Psi_\alpha - (\Delta_{\Phi, \Psi})^{1/2} x\Psi \| \to 0 .
\end{aligned}$$

Since $M\ s^M(\Psi)\Psi + (\mathbb{1} - s^{M'}(\Psi))H$ is a core for $(\Delta_{\Phi, \Psi})^{1/2}$, the vectors

$$(\mathbb{1} + (\Delta_{\Phi, \Psi})^{1/2}) x\Psi, \quad x \in M s^M(\Psi)$$

are dense in $s^{M'}(\Psi)H$. Since

$$\| [\mathbb{1} + (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2}]^{-1} \| \leqq 1$$

is uniformly bounded, we obtain

$$\lim [\mathbb{1} + (\Delta_{\Phi_\alpha, \Psi_\alpha})^{1/2}]^{-1} s^{M'}(\Psi) = [\mathbb{1} + (\Delta_{\Phi, \Psi})^{1/2}]^{-1} s^{M'}(\Psi).$$

The rest of the proof is standard. For $r > 0$ and $\Delta_\alpha = \Delta_\alpha^* \geqq 0$,

$$(r + \Delta_\alpha)^{-1} = R_r(\Delta_\alpha)(\mathbb{1} + \Delta_\alpha)^{-1} \tag{4.4}$$

with

$$R_r(\Delta_\alpha) = \{ \mathbb{1} + (r-1)(\mathbb{1} + \Delta_\alpha)^{-1} \}^{-1},$$

$$\| R_r(\Delta_\alpha) \| \leqq \max\{1, r^{-1}\}.$$

If $\Delta = \Delta^* \geqq 0$, $\lim (\mathbb{1} + \Delta_\alpha)^{-1} s = (\mathbb{1} + \Delta)^{-1} s$ for a projection s commuting with Δ, then the formula

$$\begin{aligned}
(r + \Delta_\alpha)^{-1} - (r + \Delta)^{-1} &= R_r(\Delta_\alpha) \{ (\mathbb{1} + \Delta_\alpha)^{-1} - (\mathbb{1} + \Delta)^{-1} \} \\
&\quad - R_r(\Delta_\alpha)(r-1) \{ (\mathbb{1} + \Delta_\alpha)^{-1} - (\mathbb{1} + \Delta)^{-1} \} R_r(\Delta)(\mathbb{1} + \Delta)^{-1}
\end{aligned}$$

implies

$$\lim\{(r+\Delta_\alpha)^{-1}-(r+\Delta)^{-1}\}s=0,$$

where the convergence is uniform if r is restricted to any compact subset of $(0, \infty)$. By applying this result to $\Delta_\alpha=(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}$, $\Delta=(\Delta_{\Phi,\Psi})^{1/2}$ and $s=s^{M'}(\Psi)$, we obtain the Lemma. Q.E.D.

Proof of Theorem 3.7 (1). We divide our proof into several steps. Obviously we may omit those α for which $s(\psi_\alpha)\geqq s(\phi_\alpha)$ does not hold out of our consideration so that we may assume $s(\psi_\alpha)\geqq s(\phi_\alpha)$ for all α without loss of generality.

(a) *The case where* ψ *is faithful*: Due to $s(\psi)=\mathbf{1}$, we have $s^{M'}(\Psi)=Js^M(\Psi)J=\mathbf{1}$. Hence (4.2) and Lemma 4.1 imply

$$\lim\int_\varepsilon^L \mathrm{d}r(\Phi_\alpha,\{(\mathbf{1}+r)^{-1}-[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha) \tag{4.5}$$

$$=\int_\varepsilon^L \mathrm{d}r(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Phi,\Psi})^{1/2}]^{-1}\}\Phi)$$

for all $0<\varepsilon<L<\infty$. (Note that

$$\|[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\|\leqq r^{-1}$$

is uniformly bounded.)

We also have the following estimates:

$$\int_0^\varepsilon \mathrm{d}r\left|(\Phi_\alpha,\{(1+r)^{-1}-[r+(\Delta_{\Phi_\alpha,\Psi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)\right| \tag{4.6}$$

$$=\int_0^\varepsilon \mathrm{d}r\left|\int_0^\infty(1+r)^{-1}(1+r\lambda^{-1/2})^{-1}(1-\lambda^{-1/2})\,\mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)\right|$$

$$\leqq\int_0^\varepsilon \mathrm{d}r\int_0^\infty \max(1,\lambda^{-1})\,\mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)$$

$$\leqq\varepsilon(\phi_\alpha(\mathbf{1})+\psi_\alpha(\mathbf{1}))$$

due to (3.2), where $E_\lambda^{\Phi_\alpha,\Psi_\alpha}$ is the spectral projection of $\Delta_{\Phi_\alpha,\Psi_\alpha}$.

$$\int_L^\infty \mathrm{d}r\left|\int_0^1\{(1+r)^{-1}-(r+\lambda^{1/2})^{-1}\}\,\mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)\right| \tag{4.7}$$

$$=\int_L^\infty \mathrm{d}r\int_0^1(1+r)^{-1}(1+r\lambda^{-1/2})^{-1}(\lambda^{-1/2}-1)\,\mathrm{d}(\Phi_\alpha, E_\lambda^{\Phi_\alpha,\Psi_\alpha}\Phi_\alpha)$$

$$\leqq \int_{L}^{\infty} \mathrm{d}r(1+r)^{-1}r^{-1} \int_{0}^{1} \mathrm{d}(\Phi_{\alpha}, E_{\lambda}^{\Phi_{\alpha}, \Psi_{\alpha}} \Phi_{\alpha})$$

$$\leqq \log(1+L^{-1}) \phi_{\alpha}(\mathbf{1}).$$

Finally

$$(4.8) \qquad \int_{L}^{\infty} \mathrm{d}r \int_{1}^{\infty} \{(1+r)^{-1} - (r+\lambda^{1/2})^{-1}\} \mathrm{d}(\Phi_{\alpha}, E_{\lambda}^{\Phi_{\alpha}, \Psi_{\alpha}} \Phi_{\alpha}) \geqq 0.$$

Hence

$$(4.9) \qquad \liminf \int_{0}^{\infty} \mathrm{d}r(\Phi_{\alpha}, \{(1+r)^{-1} - [r+(\Delta_{\Phi_{\alpha}, \Psi_{\alpha}})^{1/2}]^{-1}\} \Phi_{\alpha})$$

$$\geqq \int_{\varepsilon}^{L} \mathrm{d}r(\Phi, \{(1+r)^{-1} - [r+(\Delta_{\Phi, \Psi})^{1/2}]^{-1}\} \Phi)$$

$$-\varepsilon(\phi(\mathbf{1}) + \psi(\mathbf{1})) - \{\log(1+L^{-1})\} \phi(\mathbf{1}).$$

We now use the following formula, which holds if $s(\psi) \geqq s(\phi)$.

$$(4.10) \qquad S(\psi/\phi) = 2\int_{0}^{\infty} \left(\int_{0}^{\infty} \{(1+r)^{-1} - (r+\lambda^{1/2})^{-1}\} \mathrm{d}r \right) \mathrm{d}(\Phi, E_{\lambda}^{\Phi, \Psi} \Phi)$$

$$= 2\int_{0}^{\infty} \mathrm{d}r(\Phi, \{(1+r)^{-1} - [r+(\Delta_{\Phi, \Psi})^{1/2}]^{-1}\} \Phi),$$

where the change of the order of r- and λ- integrations is allowed because the integral is definite in the Lebesgue sense (finite or $+\infty$) due to (3.2).

By taking the limit $\varepsilon \to +0$ and $L \to +\infty$ and by substituting (4.10) and the same formula for the pair $\psi_{\alpha}, \phi_{\alpha}$, we obtain Theorem 3.7 (1) for this case.

(b) *The case where ϕ_{α} is independent of α*: By (3.3) and by the same computation as (4.10), we obtain

$$(4.11) \qquad S(\psi_{\alpha}/\phi) = -2\int_{0}^{\infty} \mathrm{d}r(\Phi, \{(1+r)^{-1} - [r+(\Delta_{\Psi_{\alpha}, \Phi})^{1/2}]^{-1}\} \Phi)$$

where the boundedness

$$(4.12) \qquad \int_{0}^{\infty} \lambda \mathrm{d}(\Phi, E_{\lambda}^{\Psi_{\alpha}, \Phi} \Phi) = \|(\Delta_{\Psi_{\alpha}, \Phi})^{1/2} \Phi\|^{2} = \psi_{\alpha}(s(\phi)) < \infty$$

guarantees the definiteness of the integral in (4.11). (Note that $s(\psi_{\alpha}) \geqq s(\phi_{\alpha}) = s(\phi)$.)

By Lemma 4.1 and by the same argument as the Case (a), we obtain

(4.13) $$\liminf S(\psi_\alpha/\phi) \geqq -2\int_0^\infty dr(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi).$$

Since $(\Delta_{\Psi,\Phi})^{1/2}$ commutes with $s^M(\Psi)=Js^{M'}(\Psi)J$, the inner product in (4.13) is the sum of contributions from the expectation values in $(\mathbf{1}-s^M(\Psi))\Phi$ and $s^M(\Psi)\Phi$. The first one is given by

$$-2\int_0^\infty dr(\Phi,\{(1+r)^{-1}-r^{-1}\}(\mathbf{1}-s^M(\Psi))\Phi)=+\infty$$

if

$$(\Phi,(\mathbf{1}-s^M(\Psi))\Phi)=\phi(\{\mathbf{1}-s(\psi)\})>0,$$

i.e. if $s(\psi)\geqq s(\phi)$ does not hold. The second one is either finite or $+\infty$ by (4.12). Hence if $s(\psi)\geqq s(\phi)$ does not hold, then

(4.14) $$\lim S(\psi_\alpha/\phi)=+\infty=S(\psi/\phi).$$

If $s(\psi)\geqq s(\phi)$ holds, then (4.13) already proves Theorem 3.7 (1) for the present case.

(c) *General case*: Let ω be a normal faithful state. For $\varepsilon>0$, we obtain

$$\liminf S(\psi_\alpha+\varepsilon\omega/\phi_\alpha)\geqq S(\psi+\varepsilon\omega/\phi)$$

by the Case (a). By Theorem 3.6 (4),

$$S(\psi_\alpha+\varepsilon\omega/\phi_\alpha)\leqq S(\psi_\alpha/\phi_\alpha).$$

Hence

$$\liminf S(\psi_\alpha/\phi_\alpha)\geqq S(\psi+\varepsilon\omega/\phi).$$

By taking the limit $\varepsilon\to+0$ and using the Case (b), we obtain Theorem 3.7 (1) for the general case.

Proof of Theorem 3.7 (2). If $\omega'\geqq\lambda^{-1}\omega$ for $\lambda>0$, then (3.7) implies

$$(\Delta_{\Omega',\Omega}+r)^{-1}\leqq(\lambda^{-1}\Delta_{\Omega,\Omega}+r)^{-1}.$$

Due to the identity

$$(r+\rho^{1/2})^{-1}=\pi^{-1}\int_0^\infty(\rho+x)^{-1}(x+r^2)^{-1}x^{1/2}dx\,,\quad r>0\,,$$

for a positive self-adjoint ρ, this implies

$$(r+(\Delta_{\Omega',\Omega})^{1/2})^{-1}\leqq(r+\lambda^{-1/2}(\Delta_{\Omega,\Omega})^{1/2})^{-1}.$$

Hence

$$\begin{aligned}\omega(\mathbf{1})&\geqq(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&\geqq\omega(\mathbf{1})\{(1+r)^{-1}-(\lambda^{-1/2}+r)^{-1}\}\\&=\omega(\mathbf{1})(1+r)^{-1}(1+\lambda^{1/2}r)^{-1}(1-\lambda^{1/2}).\end{aligned}$$

Therefore

$$\text{(4. 15)}\quad \begin{aligned}-\varepsilon\omega(\mathbf{1})&\leqq-\int_0^\varepsilon \mathrm{d}r(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&\leqq\omega(\mathbf{1})\log\{(1+\varepsilon)(1+\lambda^{1/2}\varepsilon)^{-1}\}\end{aligned}$$

for $\varepsilon>0$. We also have for $L>0$

$$\text{(4. 16)}\quad \begin{aligned}\omega(\mathbf{1})\log(1+L^{-1})&=-\int_L^\infty(\Omega,\{(1+r)^{-1}-r^{-1}\}\Omega)\\&\geqq-\int_L^\infty \mathrm{d}r(\Omega,\{(1+r)^{-1}-[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\}\Omega)\\&=-\int_L^\infty \mathrm{d}r(\Omega,(1+r)^{-1}[r+(\Delta_{\Omega',\Omega})^{1/2}]^{-1}\{(\Delta_{\Omega',\Omega})^{1/2}-\mathbf{1}\}\Omega)\\&\geqq-\int_L^\infty(1+r)^{-1}r^{-1}\mathrm{d}r\|(\Delta_{\Omega',\Omega})^{1/2}\Omega\|^2\\&=-\omega'(\mathbf{1})\log(1+L^{-1}).\end{aligned}$$

where the last inequality is obtained by using the spectral decomposition $\Delta_{\Omega',\Omega}=\int\lambda\mathrm{d}E_\lambda$ and majorizing $(r+\lambda^{1/2})^{-1}(\lambda^{1/2}-1)$ by $r^{-1}\lambda$ for $0\leqq\lambda$. Since

$$\begin{aligned}\lim\int_\varepsilon^L &\mathrm{d}r(\Phi_\alpha,\{(1+r)^{-1}-[r+(\Delta_{\Psi_\alpha,\Phi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)\\&=\int_\varepsilon^L \mathrm{d}r(\Phi,\{(1+r)^{-1}-[r+(\Delta_{\Psi,\Phi})^{1/2}]^{-1}\}\Phi),\end{aligned}$$

the estimates (4. 15) and (4. 16) for $(\omega',\omega)=(\psi_\alpha,\phi_\alpha)$ and for $(\omega',\omega)=(\psi,\phi)$ yield

$$(4.17)\quad \lim \int_0^\infty dr(\Phi_\alpha, \{(1+r)^{-1} - [r + (\Delta_{\Psi_\alpha, \Phi_\alpha})^{1/2}]^{-1}\}\Phi_\alpha)$$

$$= \int_0^\infty dr(\Phi, \{(1+r)^{-1}[r + (\Delta_{\Psi, \Phi})^{1/2}]^{-1}\}\Phi).$$

Since $\lambda\psi_\alpha \geqq \phi_\alpha$ and its consequence $\lambda\psi \geqq \phi$ imply $s(\psi_\alpha) \geqq s(\phi_\alpha)$ and $s(\psi) \geqq s(\phi)$, the equations (4.17) and an expression of the form (4.11) for $s(\psi_\alpha/\phi_\alpha)$ and $s(\psi/\phi)$ imply Theorem 3.7 (2).

Proof of Theorem 3.7 (3). This follows from Theorem 3.7 (1) and Theorem 3.6 (4). Q.E.D.

Remark 4.2. The argument leading to (4.14) implies that the formula

$$(4.18)\quad S(\psi/\phi) = -2\int_0^\infty dr(\Phi, \{(1+r)^{-1} - [r + (\Delta_{\Psi, \Phi})^{1/2}]^{-1}\}\Phi),$$

which is used in (4.11) for the case $s(\psi) \geqq s(\phi)$, holds for a general pair ψ and ϕ (even if $s(\psi) \geqq s(\phi)$ does not hold). this is not the case for the formula of the form (4.10).

Proof of Theorem 3.9 (1). Let ω_0 be a faithful state, $\omega = \omega_0 + \psi + \phi$, and $1 > \varepsilon > \mu > 0$. The proof of Lemma 3 in [1] (without the assumption $\psi \leqq k\phi$ there) implies

$$(4.19)\quad \liminf S(E_{N_\alpha}\psi_\varepsilon / E_{N_\alpha}\phi_\mu) \geqq S(\psi_\varepsilon/\phi_\mu)$$

where

$$\psi_\varepsilon = (1-\varepsilon)\psi + \varepsilon\omega, \quad \phi_\mu = (1-\mu)\phi + \mu\omega.$$

By the convexity (Theorem 3.8 (1)),

$$(4.20)\quad S(E_{N_\alpha}\psi_\varepsilon / E_{N_\alpha}\phi_\mu)$$

$$\leqq (1-\mu) S(E_{N_\alpha}\psi_\varepsilon / E_{N_\alpha}\phi) + \mu S(E_{N_\alpha}\psi_\varepsilon / E_{N_\alpha}\omega).$$

By Theorem 3.6 (4) and (3), we have

$$(4.21)\quad S(E_{N_\alpha}\psi_\varepsilon / E_{N_\alpha}\omega) \leqq S(E_{N_\alpha}(\varepsilon\omega) / E_{N_\alpha}\omega)$$

$$= -\omega(\mathbf{1}) \log \varepsilon < \infty.$$

By Theorem 3.7 (2)

(4. 22) $$\lim_{\mu\to 0} S(\psi_\varepsilon/\phi_\mu) = S(\psi_\varepsilon/\phi).$$

The formulas (4. 20), (4. 21) and (4. 22) imply in the limit $\mu\to+0$

(4. 23) $$\liminf S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi) \geqq S(\psi_\varepsilon/\phi).$$

By Theorem 3. 7 (3),

(4. 24) $$\lim_{\varepsilon\to 0} S(\psi_\varepsilon/\phi) = S(\psi/\phi).$$

By Theorem 3. 6 (4),

(4. 25) $$\begin{aligned} S(E_{N_\alpha}\psi_\varepsilon/E_{N_\alpha}\phi) &\leqq S(E_{N_\alpha}(1-\varepsilon)\psi/E_{N_\alpha}\phi) \\ &= S(E_{N_\alpha}\psi/E_{N_\alpha}\phi) - \phi(\mathbf{1})\log(1-\varepsilon). \end{aligned}$$

The formulas (4. 23), (4. 24) and (4. 25) imply in the limit $\varepsilon\to+0$ Theorem 3. 9 (1).

Proof of Theorem 3. 9 (2). This follows from Theorem 3. 9 (1) and Theorem 3. 8 (2) (γ). Q.E.D.

Proof of Theorem 3. 10. First consider the case where ϕ is faithful. Then Ω given by (2. 8) is cyclic and separating for M. From the definition of the perturbed state and the expression (2. 10), we obtain

(4. 26) $$\Psi(h)\otimes e_{11} + \Phi\otimes e_{22} = \Omega(h\otimes u_{11}).$$

By (4. 13) of [8], we have

(4. 27) $$\log \Delta_{\Omega(h\otimes u_{11})} = \log \Delta_\Omega + h\otimes u_{11} - j(h)\otimes J_K u_{11} J_k .$$

Here $j(h)$ denotes JhJ. By (2. 10), we obtain

(4. 28) $$\log \Delta_{\Psi(h),\Phi} = \log \Delta_{\Psi,\Phi} + h ,$$

(4. 29) $$\log \Delta_{\Phi,\Psi(h)} = \log \Delta_{\Phi,\Psi} - j(h).$$

By (3. 3), for example, we obtain Theorem 3. 10 for the present case of a faithful ϕ.

For the general case, we apply the result just proved to

$$\phi_\varepsilon = (1-\varepsilon)\phi + \varepsilon\psi, \quad \varepsilon>0,$$

which is faithful:

(4. 30) $$S(\psi^h/\phi_\varepsilon) = -(1-\varepsilon)\phi(h) - \varepsilon\psi(h) + S(\psi/\phi_\varepsilon).$$

From the convexity of the relative entropy, we obtain

$$S(\psi^h/\phi_\varepsilon) \leqq (1-\varepsilon)S(\psi^h/\phi) - \varepsilon\psi(h).$$

Combining the limit $\varepsilon \to +0$ of this relation with Theorem 3. 7 (1), we obtain

(4. 31) $$\lim_{\varepsilon\to+0} S(\psi^h/\phi_\varepsilon) = S(\psi^h/\phi).$$

For $h=0$, we have the same equation for ψ. Hence the first equation of Theorem 3. 10 follows from (4. 30). The second equation of Theorem 3. 10 is trivially true for a non-faithful ϕ because both sides of the equation is then $+\infty$.

§ 5. Relative Entropy of States of C^*-Algebras

For two positive linear functionals ψ and ϕ of a C^*-algebra $\mathfrak{A}$, we define the relative entropy $S(\psi/\phi)$ by

(5. 1) $$S(\psi/\phi) \equiv S(\tilde{\psi}/\tilde{\phi})$$

where $\tilde{\psi}$ and $\tilde{\phi}$ are the unique normal extension of ψ and ϕ to the enveloping von Neumann algebra $\mathfrak{A}''$.

If the cyclic representation π_ψ associated with ψ does not quasi-contain the cyclic representation π_ϕ associated with ϕ, then the central support of $\tilde{\psi}$ does not majorize that of $\tilde{\phi}$, hence $s(\tilde{\psi}) \geq s(\tilde{\phi})$ does not hold. Therefore

(5. 2) $$S(\psi/\phi) = +\infty$$

if π_ψ does not quasi-contain π_ϕ.

From the definition (5. 1), it follows that

(5. 3) $$S(\psi/\phi) = S(\hat{\psi}/\hat{\phi})$$

where $\hat{\psi}$ and $\hat{\phi}$ are the unique normal extension of ψ and ϕ to $M=\pi(\mathfrak{A})''$ where $\pi=\pi_\psi \oplus \pi_\phi$. If $\mathfrak{A}$ is separable, then $M=\pi(\mathfrak{A})''$ for this π has a separable predual and hence all results in previous sections apply. In particular, if $\mathfrak{A}_\alpha$ is a monotone increasing net of nuclear C^*-subalgebras of $\mathfrak{A}$ generating $\mathfrak{A}$, then

(5. 4) $$\lim S(E_{\mathfrak{A}_\alpha}\psi/E_{\mathfrak{A}_\alpha}\phi)=S(\psi/\phi).$$

This implies the result in [2] that if

$$\sup_\alpha S(E_{\mathfrak{A}_\alpha}\psi/E_{\mathfrak{A}_\alpha}\phi)<\infty,$$

then π_ψ quasi-contains π_ϕ.

If $\mathfrak{A}$ is separable, then the restriction of the envelopping von Neumann algebra $\mathfrak{A}''$ to a direct sum of a denumerable number of cyclic representations of $\mathfrak{A}$ has a faithful normal state. Hence Theorems 3. 6, 3. 8, and 3. 9 as well as Theorem 3. 7 for sequences are valid for positive linear functionals of C^*-algebras.

If ψ is a positive linear functional of a C^*-algebra $\mathfrak{A}$ such that the corresponding cyclic vector Ψ for the associated cyclic representation π_ψ of $\mathfrak{A}$ is separating for the weak closure $\pi_\psi(\mathfrak{A})''$, then the perturbed state ψ^h for $h=h^*\in\mathfrak{A}$ is defined by

(5. 5) $$\psi^h(a)=(\Psi[\pi_\psi(h)],\pi_\psi(a)\Psi[\pi_\psi(h)]),\quad a\in\mathfrak{A}.$$

For such ψ, Theorem 3. 10 holds for C^*-algebras.

§ 6. Conditional Entropy

Let $\mathfrak{A}$ be a UHF algebra with an increasing suquence of finite dimensional factors $\mathfrak{A}_n$ generating $\mathfrak{A}$. Let $\mathfrak{A}^c_{m,n}$ be the relative commutant of $\mathfrak{A}_n$ in $\mathfrak{A}_m$. The conditional entropy $\tilde{S}_n(\phi)$ of a positive linear functional ϕ of $\mathfrak{A}$ is defined by

(6. 1) $$\tilde{S}_n(\phi)=\lim_{m\to\infty}(S(E_{\mathfrak{A}_m}\phi)-S(E_{\mathfrak{A}^c_{m,n}}\phi))$$

where

$$S(\psi)=-\psi(\log\rho_\psi)$$

for a positive linear functional ψ of a finite dimensional factor $\mathfrak{M}_k$ and ρ_ψ is the density matrix of ψ defined by

$$\psi(a)=\tau(\rho_\psi a),\quad a\in\mathfrak{M}_k$$

with the unique trace state τ of $\mathfrak{M}_k$. ([3])

Let $\mathfrak{A}^c_{\cdot,n}$ be the relative commutant of $\mathfrak{A}_n$ in $\mathfrak{A}$, ω be the restriction of ϕ to $\mathfrak{A}^c_{\cdot,n}$ and ω' be any positive linear functional on $\mathfrak{A}^c_{\cdot,n}$. Then

$$(6.2)\quad S(E_{\mathfrak{A}_n}\phi)-S(E_{\mathfrak{A}^c_{m,n}}\phi)=-S(E_{\mathfrak{A}_m}(\tau_n\otimes\omega')/E_{\mathfrak{A}_m}\phi)+S(E_{\mathfrak{A}^c_{m,n}}\omega'/E_{\mathfrak{A}^c_{m,n}}\omega)$$

where τ_n is the unique trace state on $\mathfrak{A}_n$, because the density matrices for $E_{\mathfrak{A}_m}(\tau_n\otimes\omega')$ and for $E_{\mathfrak{A}^c_{m,n}}\omega'$ are the same element of $\mathfrak{A}$.

By taking the limit $m\to\infty$ and using (5.4), we obtain

$$(6.3)\quad \tilde{S}_n(\phi)=S(\omega'/\omega)-S(\tau_n\otimes\omega'/\phi).$$

Since the left hand side is finite, it follows that if either $S(\omega'/\omega)$ or $S(\tau_n\otimes\omega'/\phi)$ is finite, then both quantities are finite and (6.3) holds. This formula has been used in [3].

References

[1] Araki, H., *Publ. RIMS Kyoto Univ.* **11** (1975-76), 809-833.
[2] ———, *Commun. Math. Phys.*, **44** (1975), 1-7.
[3] ——— and Sewell, G. L., *Commun. Math. Phys.*, **52** (1977), 103-109.
[4] ———, *Pacific J. Math.*, **50** (1974), 309-354.
[5] Connes, A., *Ann. Inst. Fourier, Grenoble.*, **24**, 4 (1974), 121-155.
[6] Haagerup, U., *Math. Scand.*, **37** (1975), 271-283.
[7] Connes, A., *Ann. Scient. Ecole Norm. Sup.* 4e série, **6** (1973), 133-252.
[8] Araki, H., *Publ. RIMS Kyoto Univ.*, **9** (1973), 165-209.