Algebras and States in JT Gravity

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Abstract: We analyze the algebra of boundary observables in canonically quantised JT gravity with or without matter. In the absence of matter, this algebra is commutative, generated by the ADM Hamiltonian. After coupling to a bulk quantum field theory, it becomes a highly noncommutative algebra of Type II_∞ with a trivial center. As a result, density matrices and entropies on the boundary algebra are uniquely defined up to, respectively, a rescaling or shift. We show that this algebraic definition of entropy agrees with the usual replica trick definition computed using Euclidean path integrals. Unlike in previous arguments that focused on $\mathcal{O}(1)$ fluctuations to a black hole of specified mass, this Type II_∞ algebra describes states at all temperatures or energies. We also consider the role of spacetime wormholes. One can try to define operators associated with wormholes that commute with the boundary algebra, but this fails in an instructive way. In a regulated version of the theory, wormholes and topology change can be incorporated perturbatively. The bulk Hilbert space $\mathcal{H}_{\mathrm{bulk}}$ that includes baby universe states is then much bigger than the space of states $\mathcal{H}_{\mathrm{bdry}}$ accessible to a boundary observer. However, to a boundary observer, every pure or mixed state on $\mathcal{H}_{\mathrm{bulk}}$ is equivalent to some pure state in $\mathcal{H}_{\mathrm{bdry}}$.

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Contents

1 Introduction **1**

2 The Bulk Hilbert Space **5**
- 2.1 The Boundary Hamiltonian 5
- 2.2 The Hilbert Space of Pure JT Gravity 8
- 2.3 Including Matter Fields 13

3 The Algebra **16**
- 3.1 Warm up: Pure JT Gravity 16
- 3.2 Definition using canonical quantisation 18
- 3.3 Definition using Euclidean path integrals 23

4 Baby Universe "Operators" **31**

5 Wormhole Corrections **40**
- 5.1 Overview 40
- 5.2 The Hilbert Space From A Bulk Point Of View 42
- 5.3 The Hilbert Space From A Boundary Point of View 43
- 5.4 Relating the Boundary and the Bulk 44
- 5.5 Further Steps 50
- 5.6 Some Final Remarks 54

6 Multiple Open Universes **55**

1 Introduction

JT gravity in two dimensions [1, 2] with negative cosmological constant provides a simple and much-studied model of a two-sided black hole (for example, see [3–9]). JT gravity coupled to additional matter fields, described by a quantum field theory, has also been much studied, especially in the case that the matter theory is conformally invariant [10]. The essential simplicity of the model is retained as long as there is no direct coupling of the dilaton of JT gravity to other matter fields.

In the present article, we will study JT gravity from the point of view of understanding the algebra of observables accessible to a boundary observer living on one side of the system. It is believed that in JT gravity with or without additional matter fields, it is not possible to define a one-sided black hole Hilbert space, but it is certainly possible to define a two-sided Hilbert space $\mathcal{H}$, as studied for example in [5, 6, 8, 11–13]. We will analyze the algebra $\mathcal{A}$ of operators acting on $\mathcal{H}$ that can be defined on, say, the left boundary of the system.

The analogous problem for more complicated systems in higher dimensions has been studied recently. Those analyses have involved a large N limit and quantum fields propagating in a definite spacetime, a black hole of prescribed mass. The starting point has been a Type III$_1$ algebra of bulk quantum fields outside the black hole horizon, which can be interpreted as an algebra of single-trace boundary operators [14, 15]. Upon including in the algebra the generator of time translations, either by including certain corrections of order $1/N$ or by going to a microcanonical description, the Type III$_1$ algebra becomes a Type II$_\infty$ algebra [16, 17]. This Type III$_1$ or Type II$_\infty$ algebra describes fluctuations about a definite spacetime, namely the black hole spacetime that served as input.

The simplicity of JT gravity is such that it is possible to describe an algebra of boundary observables for JT gravity coupled to a definite QFT, without taking any sort of large N limit. One obtains an algebra $\mathcal{A}$ of Type II$_\infty$ that is equally valid for any value of the black hole temperature or mass. The trace in this algebra is the expectation value in the high temperature limit of the thermofield double state.[1] At high temperatures, the fluctuations in the bulk spacetime are small, and as in [14–17], the operators in $\mathcal{A}$ can be given a bulk interpretation. At low temperatures, the fluctuations in the bulk spacetime are large and $\mathcal{A}$ cannot be usefully approximated as an algebra of bulk operators; it has to be understood as an algebra of boundary operators.

The fact that the algebra $\mathcal{A}$ can be defined in the case of JT gravity without choosing a reference temperature means that it is "background independent." That is not the case for existing constructions of an algebra of observables outside a black hole horizon in more complicated models in higher dimensions. In those constructions, background independence is lost when one subtracts the thermal expectation value of an operator so as to get an algebra of operators that have a large N limit. In JT gravity coupled to matter, since we define the algebra without considering any large N limit for the matter system, background independence is retained.

The Type II$_\infty$ algebra $\mathcal{A}$ that describes JT gravity coupled to matter is a "factor," meaning that its center consists only of c-numbers. Accordingly, $\mathcal{A}$ has a trace that is uniquely determined up to an overall multiplicative constant. A multiplicative constant in the trace leads to an additive constant in the entropy, so a state of the algebra $\mathcal{A}$ has an entropy that is uniquely defined up to a state-independent additive constant. By contrast, in JT gravity without matter, the algebra of boundary observables is commutative – generated by the ADM Hamiltonian. Therefore, in the absence of matter, the algebraic structure alone does not determine a unique trace or an appropriate definition of entropy. We will see that when matter is present so that the algebra is of Type II$_\infty$, the entropy of a state of the Type II$_\infty$ algebra agrees with the entropy computed via Euclidean path integrals [19–21] up to an overall additive constant. Similar results were obtained previously in analyses based on large N limits [17]. In contrast, previous attempts at understanding entanglement entropy in canonically quantised JT gravity focused on JT gravity without matter. As a result, they relied on the introduction of additional ingredients into the

[1] This role of the high temperature limit has also been noted in the context of a double-scaled version of the SYK model [13]; in that context, the algebra is of Type II$_1$. A description of double-scaled SYK in which the high temperature limit is conveniently accessible had been developed in [18].

theory, such as the defect operators considered in [5, 12], that were "fine-tuned" to match the Euclidean path integral results.

The bulk Hilbert space $\mathcal{H}_{\rm bulk}$ and the algebra $\mathcal{A}$ can be naturally-defined in a "no-wormhole" version of the theory in which the spacetime topology is assumed to be a Lorentzian strip (or equivalently a disc in Euclidean signature), and this is quite natural for everything that we have said up to this point. However, it is also interesting to ask what happens if we incorporate wormholes and baby universes. In pure JT gravity, there is no difficulty in studying wormhole contributions order by order in the genus of spacetime or equivalently in e^{-S}, where S is the entropy. (The expansion in powers of e^{-S} will break down at low temperatures.) One can even understand the theory nonperturbatively via a dual matrix model [23]. When the theory is coupled to matter, however, the perturbative wormhole contributions diverge because the negative matter Casimir energy in a closed universe leads to a divergent contribution from small wormholes. In fact this divergence plays a crucial and illustrative role in ensuring the consistency of the "no-wormhole" story described above. It does so by avoiding the presence of "baby universe operators," similar to those in [22], whose eigenvalues would be classical α-parameters [24, 25].

In a more complete theory, we might expect that the Casimir divergence should be regulated. In the SYK model, for example, the divergence is suspected to be regulated by something similar to the Hawking-Page-like phase transition described in [26]; see Section 6.1 of [23] for discussion on this point. As a result, we proceed somewhat formally and attempt to understand what happens to the boundary algebras and the Hilbert space in such a regulated theory. The analysis of the algebra $\mathcal{A}$ gives no major surprises: it is corrected order by order in the wormhole expansion, but remains an algebra of Type II_∞. The analysis of the Hilbert space is more subtle and involves an interesting difference between the no-wormhole theory and the theory with wormholes included. With or without wormholes, a Hilbert space $\mathcal{H}_{\rm bdry}$ can be defined from a boundary point of view by first introducing states that have a reasonable Euclidean construction, then using the bulk path integral to compute inner products among these states, and finally dividing out null vectors and taking a completion to get a Hilbert space. The inner products that enter this construction have wormhole corrections, but wormholes do not affect the "size" of $\mathcal{H}_{\rm bdry}$. On the other hand, from a bulk point of view, once we include wormholes, to define a Hilbert space we have to include closed "baby universes." The resulting Hilbert space $\mathcal{H}_{\rm bulk}$ is then much "bigger" than it would be in the absence of wormholes. There is a fairly simple natural definition of $\mathcal{H}_{\rm bdry}$ and a fairly simple natural definition of $\mathcal{H}_{\rm bulk}$, but it is less obvious how to relate them. We make a sort of gauge choice that enables us to define a map $\mathcal{W}:\mathcal{H}_{\rm bdry}\to\mathcal{H}_{\rm bulk}$ that preserves inner products, embedding $\mathcal{H}_{\rm bdry}$ as a rather "small" subspace of $\mathcal{H}_{\rm bulk}$. States in $\mathcal{H}_{\rm bulk}$ that are orthogonal to $\mathcal{W}(\mathcal{H}_{\rm bdry})$ are inaccessible to a boundary observer. The map $\mathcal{W}:\mathcal{H}=\mathcal{H}_{\rm bdry}\to\mathcal{H}_{\rm bulk}$ is awkward to describe explicitly even for states that have a simple Euclidean construction. This map is likely far more difficult to describe for states that do not have such a simple construction – for example, states that arise from Lorentz signature time evolution starting from states with a simple Euclidean construction.

One of the main results of our study of wormholes is to learn that, from the point of view of the boundary observer, at least to all orders in e^{-S} (since our analysis is based on an expansion in this parameter), any pure or mixed state on the bulk Hilbert space $\mathcal{H}_{\rm bulk}$ is equivalent to a pure state in the much smaller Hilbert space $\mathcal{H}_{\rm bdry}$. Classically, one might describe this by saying that although $\mathcal{H}_{\rm bulk}$ is much bigger than $\mathcal{H}_{\rm bdry}$, the extra degrees of freedom in $\mathcal{H}_{\rm bulk}$ are beyond the observer's horizon.

Another generalization is as follows. Instead of a world with a single open universe component and possible closed baby universes, we can consider a world with two open universe components or in general any number of them, plus baby universes. In the absence of wormholes, this adds nothing essentially new: a Hilbert space for two open universes would be trivially constructed from single-universe Hilbert spaces. With wormholes included, distinct open universes can interact with each other via wormhole exchange. However, we can ask the following question: can an observer with access to only one asymptotic boundary of spacetime know how many other boundaries there are? We show that the answer to this question is "no," at least to all orders in e^{-S}, in the following sense. Let $\mathcal{H}_{\rm bdry}$ be the boundary Hilbert space for the case of a single open universe component (and any number of closed universes), and let $\mathcal{A}$ be the algebra of boundary operators acting on $\mathcal{H}_{\rm bdry}$. The same algebra $\mathcal{A}$ also acts on the bulk Hilbert space $\mathcal{H}_{{\rm bulk},[n]}$ with any number n of open universe components (and, again, any number of closed universes), and every pure or mixed state on $\mathcal{H}_{{\rm bulk},[n]}$ is equivalent, for a boundary observer, to some pure state in $\mathcal{H}_{\rm bdry}$. Classically, one would interpret this by saying that an observer at one asymptotic end has no way to know how many other asymptotic ends there are because they are all beyond a horizon. Quantum mechanically, that language does not apply in any obvious way but the conclusion is valid.

In section 2, we review aspects of JT gravity and discuss from a bulk point of view the Hilbert space of JT gravity coupled to a quantum field theory. In section 3, we construct the algebra $\mathcal{A}$ of operators accessible to an observer outside the horizon. We define this algebra both directly within the canonically quantised theory, and via a natural alternative definition using Euclidean path integrals, which we argue is equivalent. This equivalence justifies the use of Euclidean path integrals to compute entropies in the context of JT gravity with matter. In section 4, we attempt to define "baby universe operators" that would commute with the boundary algebras, and show that this fails in an instructive fashion. In section 5, we consider wormhole corrections both to the Hilbert space constructed in section 2 and to the algebra constructed in section 3. As already noted, once wormholes are included, the Hilbert space that is natural from a boundary point of view is a "small" and difficult to characterize subspace of the Hilbert space that is natural from a bulk point of view. In section 6, we consider a further generalization to a spacetime with multiple asymptotic boundaries. As already explained, a primary conclusion of studying these generalizations is to learn that they are undetectable by an observer at infinity in one asymptotic region.

2 The Bulk Hilbert Space

In this section, we first review some aspects of JT gravity – focussing on the bare minimum needed for the present article – and then we discuss from a bulk point of view the Hilbert space of pure JT gravity and of JT gravity coupled to a quantum field theory.

2.1 The Boundary Hamiltonian

The action of JT gravity with negative cosmological constant on a spacetime M can be written, in the notation of [8, 12], as

$$I_{JT} = \int_M \mathrm{d}^2x\sqrt{-g}\phi(R+2) + 2\int_{\partial M} \mathrm{d}t\sqrt{|\gamma|}\phi(K-1) + \cdots\,, \tag{2.1}$$

where g is the bulk metric with curvature scalar R, γ is the induced metric on the boundary, K is the extrinsic curvature of the boundary, and we have omitted a topological invariant related to the classical entropy S_0. Upon integrating first over ϕ to impose the equation of motion $R+2=0$, with a boundary condition that fixes the boundary value of ϕ, the action reduces to

$$I_{\partial M} = 2\int_{\partial M} \mathrm{d}t\sqrt{|\gamma|}\phi(K-1). \tag{2.2}$$

The condition $R+2=0$ implies that M is locally isomorphic to a portion of AdS_2, a homogeneous manifold of constant curvature -2. AdS_2 is the universal cover of what we will call $\mathrm{AdS}_2^{(0)}$, namely the quadric $X^2+Y^2-Z^2=1$ with metric $\mathrm{d}s^2=-\mathrm{d}X^2-\mathrm{d}Y^2+\mathrm{d}Z^2$. $\mathrm{AdS}_2^{(0)}$ has an action of $SL(2,\mathbb{R})$ generated by vector fields

$$\begin{aligned} j_1 &= X\partial_Y - Y\partial_X \\ j_2 &= Y\partial_Z + Z\partial_Y \\ j_3 &= -X\partial_Z - Z\partial_X, \end{aligned} \tag{2.3}$$

satisfying $[j_a, j_b] = \epsilon_{ab}^c j_c$, where the metric on the Lie algebra is $\eta_{ab} = \mathrm{diag}(-1,1,1)$. Coordinates T,σ with

$$\begin{aligned} X &= \cos T\cosh\sigma \\ Y &= \sin T\cosh\sigma \\ Z &= \sinh\sigma \end{aligned} \tag{2.4}$$

give a useful parametrization of the universal cover AdS_2. In these coordinates, the metric is

$$\mathrm{d}s^2 = \mathrm{d}\sigma^2 - \cosh^2\sigma\,\mathrm{d}T^2, \qquad -\infty<\sigma,T<\infty. \tag{2.5}$$

The vector fields (2.3) on $\mathrm{AdS}_2^{(0)}$ lift to vector fields on AdS_2 that generate an action of $\widetilde{SL}(2,\mathbb{R})$, the universal cover of $SL(2,\mathbb{R})$.

AdS_2 has a "right" conformal boundary at $\sigma\to+\infty$ and a "left" conformal boundary at $\sigma\to-\infty$. In studies of JT gravity, M is usually taken to be "almost all" of AdS_2 [3, 4]. This is achieved as follows. First of all, the left and right boundaries could be defined simply by functions $\sigma_L(T)$, $\sigma_R(T)$. However, in "nearly AdS_2 holography," one assumes that the boundary is parameterised by a distinguished parameter t, the time of the boundary quantum mechanics, and one parametrizes the right boundary curve by functions $\sigma_R(t)$, $T_R(t)$, and similarly for the left boundary. To get nearly AdS_2 spacetime, one further imposes the boundary conditions

$$\gamma_{tt} = -\frac{1}{\epsilon^2}, \quad \phi|_{\partial M} = \frac{\varphi_b}{\epsilon}, \tag{2.6}$$

with constant φ_b and with very small ϵ. For small ϵ, the condition $\gamma_{tt} = -1/\epsilon^2$ reduces to

$$e^{\sigma_R} = \frac{2}{\epsilon}\dot{T}_R, \quad e^{-\sigma_L} = \frac{2}{\epsilon}\dot{T}_L, \tag{2.7}$$

where dots represent derivatives with respect to t. Thus, the left and right boundary curves lie, for small ϵ, at very large negative or positive σ, and each of them is determined by a single function $T_L(t)$ or $T_R(t)$.

It is useful to define

$$e^{\sigma_R} = \frac{2\varphi_b}{\epsilon} e^{\widetilde{\sigma}_R}, \quad e^{-\sigma_L} = \frac{2\varphi_b}{\epsilon} e^{-\widetilde{\sigma}_L}, \tag{2.8}$$

where (in view of eqn. (2.7)) $\widetilde{\sigma}_L$, $\widetilde{\sigma}_R$ remain finite for $\epsilon \to 0$. Here $\widetilde{\sigma}_L$, $\widetilde{\sigma}_R$ are renormalized length parameters, in the sense that, for $\epsilon \to 0$, the length of a geodesic between the left and right boundaries is

$$\ell = \widetilde{\sigma}_R - \widetilde{\sigma}_L + \text{constant}. \tag{2.9}$$

The constant depends only on T_L, T_R and not on $\widetilde{\sigma}_L$, $\widetilde{\sigma}_R$.

With a small calculation, one finds that for $\epsilon \to 0$, the boundary action (2.2) becomes

$$I_{\partial M} = \varphi_b \int \mathrm{d}t \left(-\dot{T}_R^2 + \left(\frac{\ddot{T}_R}{\dot{T}_R} \right)^2 \right) + \varphi_b \int \mathrm{d}t \left(-\dot{T}_L^2 + \left(\frac{\ddot{T}_L}{\dot{T}_L} \right)^2 \right), \tag{2.10}$$

which is known as the Schwarzian action because it is a linear combination of the Schwarzian derivatives $\{T_R, t\}$ and $\{T_L, t\}$.

However, there is another convenient way to describe the problem [5, 6]. One term in the boundary action (2.2) is just $-2\phi L$, where L is the length of ∂M; using the boundary condition on ϕ, this is $-\frac{2\varphi_b}{\epsilon} L$. The other term involving the integral of K can be expressed, using the Gauss-Bonnet theorem, in terms of $\int_M \mathrm{d}^2 x \sqrt{g} R$ (together with a topological invariant, the Euler characteristic of M); since $R = -2$, this is just $-2A$, with A the area of M. Thus the boundary action is

$$I_{\partial M} = \frac{2\varphi_b}{\epsilon}(A - L) + \text{constant}. \tag{2.11}$$

The area form of AdS_2 is $\sqrt{-g}\mathrm{d}\sigma\mathrm{d}T = \cosh\sigma\mathrm{d}\sigma\mathrm{d}T = \mathrm{d}(\sinh\sigma\mathrm{d}T)$, so

$$A = \int \mathrm{d}t \left(\sinh\sigma_R \frac{\mathrm{d}T_R}{\mathrm{d}t} - \sinh\sigma_L \frac{\mathrm{d}T_L}{\mathrm{d}t} \right). \tag{2.12}$$

As for the length term, the sum over all paths of length L is a random walk of that length. A random walk on a manifold describes a process of diffusion which can also be described by the heat kernel $e^{-t\Delta}$, where Δ is the Laplacian (or its Lorentz signature analog). On a general Riemannian manifold, if we take for the action the usual kinetic energy of a nonrelativistic particle, $I_{\rm kin} = \frac{1}{2}\int {\rm d}t g_{ij}\dot{x}^i\dot{x}^j$, then the corresponding Hamiltonian is $\Delta/2$, appropriate to describe diffusion. The upshot is that the L term in the action can be replaced by an action of the form $I_{\rm kin}$. The resulting action for the right boundary is then

$$I_R = \varphi_b \int {\rm d}t \left(\dot{\sigma}_R^2 - \cosh^2\sigma_R \dot{T}_R^2\right) + \frac{2\varphi_b}{\epsilon}\int {\rm d}t \sinh\sigma_R \dot{T}_R, \tag{2.13}$$

with a similar action for the left boundary. Proofs of the relationship[2] between (2.11) and (2.13) can be found in [5, 6], in part following chapter 9 of [27].

For our purposes, we will just verify that[3] (2.13) is equivalent to (2.10) in the limit $\epsilon \to 0$ (apart from an additive constant that has to be dropped from the Hamiltonian). The canonical momenta deduced from I_R are $p_{\sigma_R} = 2\varphi_b\dot{\sigma}_R$, $p_{T_R} = 2\varphi_b(-\cosh^2\sigma_R\dot{T}_R + \frac{1}{\epsilon}\sinh\sigma_R)$. The Hamiltonian is then

$$H_R = \frac{p_{\sigma_R}^2}{4\varphi_b} - \frac{1}{4\varphi_b\cosh^2\sigma_R}\left(p_{T_R} - \frac{2\varphi_b}{\epsilon}\sinh\sigma_R\right)^2. \tag{2.14}$$

Making the change of variables (2.8), where $p_{\widetilde{\sigma}_R} = p_{\sigma_R}$, the $\epsilon \to 0$ limit of the Hamiltonian comes out to be (after discarding an additive constant)

$$H_R = \frac{1}{2\varphi_b}\left(\frac{1}{2}p_{\widetilde{\sigma}_R}^2 + p_{T_R}e^{-\widetilde{\sigma}_R} + \frac{1}{2}e^{-2\widetilde{\sigma}_R}\right). \tag{2.15}$$

The action of any Hamiltonian system has a canonical form $I_{\rm can} = \int {\rm d}t(\sum_i p_i\dot{q}^i - H)$. In the present case, this is

$$I_{\rm can} = \int {\rm d}t\left(p_{\widetilde{\sigma}_R}\dot{\widetilde{\sigma}}_R + p_{T_R}\dot{T}_R - \frac{1}{2\varphi_b}\left(\frac{1}{2}p_{\widetilde{\sigma}_R}^2 + p_{T_R}e^{-\widetilde{\sigma}_R} + \frac{1}{2}e^{-2\widetilde{\sigma}_R}\right)\right). \tag{2.16}$$

Here, $I_{\rm can}$ is linear in p_{T_R}, so p_{T_R} behaves as a Lagrange multiplier setting $e^{-\widetilde{\sigma}_R} = 2\varphi_b\dot{T}_R$. After also integrating out $p_{\widetilde{\sigma}_R}$, which appears quadratically, by its equation of motion, we see that the action (2.16) is equivalent to the Schwarzian action[4] (2.10).

In terms of the variables

$$\chi_R = -\widetilde{\sigma}_R, \qquad \chi_L = \widetilde{\sigma}_L \tag{2.17}$$

[2]This relationship involves some renormalization, leading to a divergent additive constant in the Hamiltonian that will be dropped in the next paragraph.

[3]This derivation was explained to us by Z. Yang; a similar calculation in a different coordinate system can be found in [6].

[4]Introducing and simplifying the Hamiltonian has given an efficient way to do this calculation; however, one can reach the same conclusion by analyzing how the solutions of the equations of motion behave for small ϵ. We will in any case need the formula for the Hamiltonian.

used in [12], the Hamiltonian on the right boundary is

$$H_R = \frac{1}{2\varphi_b}\left(\frac{1}{2}p_{\chi_R}^2 + p_{T_R}e^{\chi_R} + \frac{1}{2}e^{2\chi_R}\right). \tag{2.18}$$

By a similar derivation, the Hamiltonian on the left boundary is[5]

$$H_L = \frac{1}{2\varphi_b}\left(\frac{1}{2}p_{\chi_L}^2 + p_{T_L}e^{\chi_L} + \frac{1}{2}e^{2\chi_L}\right). \tag{2.19}$$

The renormalized geodesic length between the left and right boundaries is

$$\ell = -\chi_R - \chi_L + \log\left(\frac{1+\cos(T_L - T_R)}{2}\right). \tag{2.20}$$

2.2 The Hilbert Space of Pure JT Gravity

The left and right boundaries of M are thus described by variables T_L, χ_L, T_R, χ_R and their canonical conjugates. Quantum mechanically, we can describe these boundaries by a Hilbert space $\mathcal{H}_0$ consisting of L^2 functions $\Psi(T_L, \chi_L, T_R, \chi_R)$.

However [5, 6, 8, 11, 12], $\mathcal{H}_0$ is not the appropriate bulk Hilbert space for JT gravity, for two reasons. One reason involves causality, and the second reason involves the gauge constraints. We will discuss causality first. Classically, one can describe a solution of JT gravity by specifying a pair of functions $T_L(t)$, $T_R(t)$ that satisfy the equations of motion derived from the Schwarzian action (2.10). Not all pairs of solutions are allowed, however; one wants the two pairs of boundaries to be spacelike separated. For the metric (2.5), the condition for this is that

$$|T_L(t) - T_R(t')| < \pi \tag{2.21}$$

for all real t, t'. Quantum mechanically, the observables $T_R(t)$ at different times are non-commuting operators that cannot be simultaneously specified; the same applies for $T_L(t)$. So we cannot directly impose the condition (2.21) at all times. Fortunately, one can check that in the classical theory it is sufficient to impose the condition (2.21) at one pair of times t, t'. As we discuss briefly below, the classical dynamics then ensure that (2.21) holds at all times so long as the the two trajectories have vanishing total $\widetilde{SL}(2,\mathbb{R})$ charge – i.e. the solution satisfies the gauge constraints. We will define the quantum theory in the same way: we impose the condition $|T_L(t) - T_R(t')| < \pi$ at some chosen times, say $t = t' = 0$, and then hope that after imposing the gauge constraints the quantum dynamics lead to a causal answer. We impose this initial condition by refining the definition of the Hilbert space $\mathcal{H}_0$ to say that it consists of L^2 functions $\Psi(T_L, \chi_L, T_R, \chi_R)$ whose support is at $|T_L - T_R| < \pi$.

Having made this definition, we then have to ask whether it leads to quantum dynamics that are consistent with causality. For JT gravity with matter, we will eventually get a fairly satisfactory answer, along the following lines. We will define algebras $\mathcal{A}_L$, $\mathcal{A}_R$ of observables on the left and right boundaries, respectively. $\mathcal{A}_L$ and $\mathcal{A}_R$ will contain,

[5] In this derivation, a minus sign in the formula (2.12) for the area is compensated by a relative minus sign in the definitions of χ_R, χ_L.

respectively, all quantum fields inserted on the left or right boundary at arbitrary values of the quantum mechanical time. The two algebras will commute with each other, and this will be a reasonable criterion for saying at the quantum level that the two boundaries are out of causal contact. For JT gravity without matter, an explanation along those lines is unfortunately not available, since there are not enough boundary observables. However the fact we end up with sensible boundary Hamiltonians on a Hilbert space constructed from wavefunctions with $|T_L - T_R| < \pi$ is itself evidence that the boundaries remain out of causal contact at all times.

Even after imposing the condition $|T_L - T_R| < \pi$, $\mathcal{H}_0$ is not the physical Hilbert space of JT gravity, because we have to impose the constraints. Since we are interested in the intrinsic geometry of M, not in how it is identified with a portion of AdS_2, we have to regard two sets of variables T_L, χ_L, T_R, χ_R that differ by the action on AdS_2 of $\widetilde{SL}(2,\mathbb{R})$ to be equivalent. In other words, we have to treat $\widetilde{SL}(2,\mathbb{R})$ as a group of constraints.

The constraint operators are

$$J_a = J_a^L + J_a^R, \tag{2.22}$$

where J_a^L and J_a^R are the generators of $\widetilde{SL}(2,\mathbb{R})$ acting on the right and left boundaries, namely

$$\begin{aligned} J_1^R &= p_{T_R} \\ J_2^R &= (\cos T_R) p_{T_R} - (\sin T_R) p_{\chi_R} + e^{\chi_R} \cos T_R + \frac{\mathrm{i}}{2} \sin T_R \\ J_3^R &= (\sin T_R) p_{T_R} + (\cos T_R) p_{\chi_R} + e^{\chi_R} \sin T_R - \frac{\mathrm{i}}{2} \cos T_R. \end{aligned} \tag{2.23}$$

and[6]

$$\begin{aligned} J_1^L &= p_{T_L} \\ J_2^L &= -(\cos T_L) p_{T_L} + (\sin T_L) p_{\chi_L} - e^{\chi_L} \cos T_L - \frac{\mathrm{i}}{2} \sin T_L \\ J_3^L &= -(\sin T_L) p_{T_L} - (\cos T_L) p_{\chi_L} - e^{\chi_L} \sin T_L + \frac{\mathrm{i}}{2} \cos T_L. \end{aligned} \tag{2.24}$$

These operators are self-adjoint and obey $[J_a^R, J_b^R] = \mathrm{i}\epsilon_{ab}{}^c J_c^R$, $[J_a^L, J_b^L] = \mathrm{i}\epsilon_{ab}{}^c J_c^L$. Here ϵ_{abc} is completely antisymmetric with $\epsilon_{123} = 1$; Lie algebra indices are raised and lowered with the metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$.

The derivation of the formulas (2.23), (2.24) can be understood as follows. The terms in J_a^L, J_b^R that are linear in the momenta give the $\sigma \to \pm\infty$ limit of the group action on the AdS_2 coordinates (σ, T) generated by the vector fields (2.3). The imaginary terms in J_a^L, J_b^R are there simply to make those operators self-adjoint. Finally, the terms proportional to e^{χ_R} and e^{χ_L} are neccessary to give the correct action on the conjugate momenta p_{χ_R}, p_{T_R} and p_{χ_L}, p_{T_L}. This action can be computed from the $\widetilde{SL}(2,\mathbb{R})$-invariant action (2.13) by taking the $\epsilon \to 0$ limit. However, it is somewhat easier to instead derive the $\widetilde{SL}(2,\mathbb{R})$

[6] The formulas for J_a^L used in [12] differ from these by $T_L \to T_L \pm \pi$, reversing the signs of J_2^L and J_3^L. We will not make this change of variables as that would make the discussion of causality less transparent.

charges in the Hamiltonian description. In this description, symmetry group generators must commute with H_L and H_R, which we have already determined. This forces the inclusion of the terms proportional to e^{χ_R}, e^{χ_L}. Actually, H_R and H_L are essentially the quadratic Casimir operators for the action of $\widetilde{SL}(2,\mathbb{R})$ on the right and left boundary degrees of freedom:

$$2\varphi_b H_R = \frac{1}{2}\left(\eta^{ab}J_a^R J_b^R - \frac{1}{4}\right), \qquad 2\varphi_b H_L = \frac{1}{2}\left(\eta^{ab}J_a^L J_b^L - \frac{1}{4}\right). \tag{2.25}$$

Before discussing how to impose these constraints at the quantum level, we first describe how they are implemented in classical JT gravity. The classical phase space procedure for dealing with a gauge symmetry is known as a symplectic quotient, and involves a two-step procedure. The starting point is a $\mathfrak{g}^*$-valued function called a "moment map" where $\mathfrak{g}$ is the Lie algebra of the gauge group and $\mathfrak{g}^*$ is its dual. This moment map should generate the gauge group action via Poisson brackets. In our case, the moment map is just the total $\widetilde{SL}(2,\mathbb{R})$ charge $J_a = J_a^L + J_a^R$, where the conserved charges J_a^L and J_a^R are given by the formulas (2.23) and (2.24) above, except that the imaginary terms can be dropped because we are in the classical limit. To take a symplectic quotient, we first consider the subspace of phase space on which the moment map is zero. To recover a symplectic manifold (i.e. a sensible phase space), we then also identify points on this constrained space that are related by the action of the gauge group. Each of these two steps reduces the phase space dimension by the dimension of the gauge group. In our case, the unconstrained phase space is eight dimensional, and the group $\widetilde{SL}(2,\mathbb{R})$ is three dimensional, so the physical phase space will be two dimensional.

The qualitative properties of a classical orbit depend on whether the Casimir $\eta^{ab}J_a^R J_b^R$ is positive, negative, or zero. If $\eta^{ab}J_a^R J_b^R < 0$, then up to an $SL(2,\mathbb{R})$ rotation, we can assume that $J_1^R \neq 0$, $J_2^R = J_3^R = 0$. The conditions $J_2^R = J_3^R = 0$ imply via eqn. (2.23) that $p_{\chi_R} = 0$ and $e^{\chi_R} = -p_{T_R}$, so that we must have $J_1^R = p_{T_R} < 0$. The $\widetilde{SL}(2,\mathbb{R})$ constraint implies that the left boundary particle has $J_a^L = -J_a^R$, and now the conditions $J_2^L = J_3^L = 0$ lead to $J_1^L = p_{T_L} = -e^{\chi_L} < 0$. But as J_1^R, J_1^L are then both negative, it is impossible to satisfy the constraint $J_1^R + J_1^L = 0$. So orbits with $\eta^{ab}J_a^R J_b^R < 0$ cannot satisfy the constraints. A similar analysis shows that the same is true of orbits with $\eta^{ab}J_a^R J_b^R = 0$.

Thus, we have to consider orbits with $\eta^{ab}J_a^R J_b^R > 0$. Any such orbit is related by $\widetilde{SL}(2,\mathbb{R})$ to one with $J_2^R > 0$ and $J_1^R = J_3^R = 0$; again, the $\widetilde{SL}(2,\mathbb{R})$ constraint requires $J_a^L = -J_a^R$. The conditions $J_1^R = J_1^L = 0$ give $p_{T_R} = p_{T_L} = 0$ and the other conditions can be solved to give

$$\begin{aligned} e^{\chi_R} &= J_2^R \cos T_R \\ e^{\chi_L} &= -J_2^L \cos T_L = J_2^R \cos T_L. \end{aligned} \tag{2.26}$$

Any orbit of this type therefore has

$$2\pi n_R - \pi/2 < T_R < 2\pi n_R + \pi/2, \qquad 2\pi n_L - \pi/2 < T_L < 2\pi n_L + \pi/2 \tag{2.27}$$

for some integers n_L, n_R. An element of the center of $\widetilde{SL}(2,\mathbb{R})$ will shift n_L, n_R by a common integer, so only the difference $n_R - n_L$ is invariant. If this difference vanishes, then $|T_L(t)-T_R(t')| < \pi$ for all t, t' and two boundaries are spacelike separated at all times. If the difference is nonzero, then $|T_L(t) - T_R(t')| > \pi$ always, and the two boundaries are timelike separated at all times. Thus it is necessary to impose a condition that the two boundaries are spacelike separated, and if this condition is imposed at one time, it remains valid for all times.

At this stage, we have reduced the phase space to a three-dimensional space parameterised by the value of J_2^R, or equivalently of the Hamiltonians $H_L = H_R = \frac{1}{4\varphi_b}(J_2^R)^2 - \frac{1}{16\varphi_b}$, along with the locations of the two boundary particles along their trajectories. To complete our analysis, we note that the gauge symmetry generator $J_2 = J_2^L + J_2^R$ preserves the gauge charges and hence preserves the two boundary trajectories. In fact (up to an energy-dependent rescaling), it generates forwards time-translation of the right boundary and backwards time-translation of the left boundary. After quotienting by this action, we obtain the final two-dimensional phase space [8] parameterised by the boundary energy $H_L = H_R$ along with the "timeshift" between the two boundary trajectories.

Let us now discuss what happens in the quantum theory. Because the constraint group $\widetilde{SL}(2,\mathbb{R})$ is non compact, imposing the constraints on quantum states is somewhat subtle. Suppose that a group G acts on a Hilbert space $\mathcal{H}_0$, with inner product $(\ ,\ )$, and one wishes to impose G as a group of constraints. In our case, $G = \widetilde{SL}(2,\mathbb{R})$ and $\mathcal{H}_0$ was defined earlier. Naively, one imposes the constraints by restricting to the G-invariant subspace of $\mathcal{H}_0$. This is satisfactory if G is compact, but if G is not compact, this procedure can be problematical because G-invariant states are typically not normalizable, so there may be few or no G-invariant states in $\mathcal{H}_0$. A procedure that often works better for a noncompact group and that has been extensively discussed in the context of gravity (see for example [28, 29]) is to define a Hilbert space of coinvariants of the G action, rather than invariants. This means that one considers any state $\Psi \in \mathcal{H}_0$ to be physical, but one imposes an equivalence relation $\Psi \cong g\Psi$ for any $g \in G$. The equivalence classes are called the coinvariants of G. G acts trivially on the space of coinvariants, since by definition Ψ and $g\Psi$ are in the same equivalence class for any $\Psi \in \mathcal{H}_0$, $g \in G$. Thus, the coinvariants are annihilated by G, even if they cannot be represented by invariant vectors in the original Hilbert space $\mathcal{H}_0$. If (as in the case of $\widetilde{SL}(2,\mathbb{R})$) the group G has a left and right invariant measure $\mathrm{d}\mu$, one can try to define an inner product on the space of coinvariants by integration over G:

$$\langle \Psi' | \Psi \rangle = \int_G \mathrm{d}\mu\ (\Psi', R(g)\Psi). \tag{2.28}$$

Here $R(g)$ is the operator by which $g \in G$ acts on $\mathcal{H}_0$. If the integral in eqn. (2.28) is convergent (as is the case for the states that will be introduced presently in eqn. (2.29)), then $\langle \Psi' | \Psi \rangle$ depends only on the equivalence classes of Ψ and Ψ', so the formula defines an inner product on the space of coinvariants and enables us to define the Hilbert space $\mathcal{H}$ of coinvariants.

The general procedure to impose constraints is really BRST quantization, or its BV generalization. Both the space of invariants and the space of coinvariants are special cases of what is natural in BRST-BV quantization. See [30] or Appendix B of [31] for background. BRST-BV quantization in general (see [32] for an introduction) permits one to define something intermediate between the space of invariants and the space of coinvariants. For example, in perturbative string theory, where one wants to impose the Virasoro generators L_n as contraints, one usually imposes a condition $L_n\Psi = 0$, $n \geq 0$, on physical states, and also an equivalence relation $\Psi \cong \Psi + L_n\chi$, $n < 0$. This means that one takes invariants of the subalgebra generated by L_n for $n \geq 0$ and coinvariants of the subalgebra generated by L_n, $n < 0$. BRST quantization generates this mixture in a natural way. Such a mixture is also natural, in general, in gauge theory and gravity.

In the case of JT gravity, such refinements are not necessary. We can just define the Hilbert space $\mathcal{H}$ of JT gravity to be the space of coinvariants of the action of $\widetilde{SL}(2,\mathbb{R})$ on $\mathcal{H}_0$. We will see that this definition leads to efficient derivations of useful results, some of which have been deduced previously by other methods. In fact, JT gravity is simple enough that it is possible, as shown in the literature, to get equivalent results, sometimes with slightly longer derivations, by working with unnormalizable $\widetilde{SL}(2,\mathbb{R})$ invariant states and correcting the inner product by formally dividing by the infinite volume of $\widetilde{SL}(2,\mathbb{R})$.

To minimize clutter, we henceforth write just T, T', χ, χ' for T_R, T_L, χ_R, χ_L. For any T, T', χ, χ' satisfying the causality constraint $|T - T'| < \pi$, there is always a unique element of $\widetilde{SL}(2,\mathbb{R})$ that sets $T = T' = 0$, $\chi = \chi'$. This means that the space of coinvariants is generated by wavefunctions of the form

$$\Psi = \delta(T)\delta(T')\delta(\chi - \chi')\psi(\chi). \tag{2.29}$$

Such wavefunctions are highly unnormalizable in the inner product of $\mathcal{H}_0$, but in the natural inner product (2.28) of the space $\mathcal{H}$ of coinvariants, we have simply

$$\langle \Psi, \Psi \rangle = \int_{-\infty}^{\infty} \mathrm{d}\chi \, \overline{\psi}\psi. \tag{2.30}$$

The form (2.29) of the wavefunction is preserved by the operator χ, acting by multiplication, along with $\widetilde{p}_\chi = p_\chi + p_{\chi'} = -\mathrm{i}(\partial_\chi + \partial_{\chi'})$. Of course, $[\widetilde{p}_\chi, \chi] = -\mathrm{i}$. In short, the physical Hilbert space $\mathcal{H}$ can be viewed as the space of square-integrable functions of χ (or χ'), and the algebra of operators acting on $\mathcal{H}$ is generated by the conjugate operators χ and $\widetilde{p}_\chi$.

Now we can evaluate the left and right Hamiltonians H_L and H_R as operators on $\mathcal{H}$. In doing so, we note that by definition any $\Psi \in \mathcal{H}$ is annihilated by the constraint operators $J_a = J_a^L + J_a^R$. This statement is just the derivative at $g = 1$ of the equivalence relation $\Psi \cong g\Psi$, $g \in \widetilde{SL}(2,\mathbb{R})$. Acting on a state of the form given in eqn. (2.29), we have

$$\begin{aligned} J_1\Psi &= (p_T + p_{T'})\Psi \\ J_2\Psi &= (p_T - p_{T'})\Psi \\ J_3\Psi &= (p_\chi - p_{\chi'})\Psi. \end{aligned} \tag{2.31}$$

So as operators on $\mathcal{H}$, p_T is equivalent to $(J_1 + J_2)/2$ and hence can be set to zero, and p_χ is equivalent to $\frac{1}{2}\widetilde{p}_\chi + \frac{1}{2}J_3$ and so can be replaced by $\frac{1}{2}\widetilde{p}_\chi$. Likewise $p_{T'}$ can be replaced by 0 and $p_{\chi'}$ by $-\frac{1}{2}\widetilde{p}_\chi$. With these substitutions, we get

$$2\varphi_b H_L = 2\varphi_b H_R = \frac{\widetilde{p}_\chi^2}{8} + \frac{e^{2\chi}}{2}. \tag{2.32}$$

From eqn. (2.20) (with $\chi_L = \chi_R = \chi$, and after absorbing a constant shift in ℓ), the renormalized length ℓ of the geodesic between the two boundaries is $\ell = -2\chi$, so alternatively

$$2\varphi_b H_L = 2\varphi_b H_R = \frac{p_\ell^2}{2} + \frac{1}{2}e^{-\ell}. \tag{2.33}$$

As noted in [12], before imposing the $\widetilde{SL}(2,\mathbb{R})$ constraints, the operators H_L, H_R are not positive-definite. On the other hand, after imposing the constraints, we have arrived at manifestly positive formulas for H_L and H_R; the negative energy states have all been removed by the constraints. This is the quantum analogue of our observation that, in classical JT gravity, orbits with $\eta^{ab}J_a^R J_b^R < 0$ cannot satisfy the constraints.

The fact that $H_L = H_R$ after imposing constraints is analogous to the fact that in higher dimensions, the ADM mass of an unperturbed Schwarzschild spacetime is the same at either end. It can be deduced directly from the relation (2.25) between the Hamiltonians and the Casimir operators. We have

$$2\varphi_b(H_R - H_L) = \frac{1}{2}\left(\eta^{ab}J_a^R J_b^R - \eta^{ab}J_a^L J_a^R\right) = \frac{1}{2}\eta^{ab}(J_a^R + J_a^L)(J_b^R - J_b^L). \tag{2.34}$$

The operator on the right hand side annihilates physical states, since any operator of the general form $\sum_a J_a X^a$, where J_a are the constraint operators and X^a are any operators, annihilates $\mathcal{H}$. Hence $H_R - H_L = 0$ as an operator on $\mathcal{H}$. Once we know this, it follows easily that H_R and H_L are positive after imposing the constraints. Since $H_R = H_L$ as operators on $\mathcal{H}$, if one of them is negative, so is the other. From eqns. (2.18) and (2.19), we see that for this to happen, p_T and $p_{T'}$ must be negative, but in this case $J_1 = p_T + p_{T'}$ is negative, contradicting the fact that J_1 annihilates physical states.

2.3 Including Matter Fields

It is pleasantly straightforward to include matter fields in this construction. As we will see, H_L and H_R remain positive.

As in many recent papers, we add to JT gravity a "matter" quantum field theory that does not couple directly to the dilaton field ϕ of JT gravity. Quantized in AdS_2, such a theory has a Hilbert space $\mathcal{H}^{\text{matt}}$. Since $\widetilde{SL}(2,\mathbb{R})$ acts on AdS_2 as a group of isometries, any relativistic field theory on AdS_2, whether conformally invariant or not, is $\widetilde{SL}(2,\mathbb{R})$-invariant. Hence the group $\widetilde{SL}(2,\mathbb{R})$ acts naturally on $\mathcal{H}^{\text{matt}}$, say with generators J_a^{matt}, obeying the $\widetilde{SL}(2,\mathbb{R})$ commutation relations.

In the context of coupling to JT gravity, the matter system should be formulated on a large piece M of AdS_2, not on all of AdS_2. However, in the limit $\epsilon \to 0$ that was reviewed in section 2.1, this distinction is unimportant because the boundary of M is, in the relevant sense, near the conformal boundary of AdS_2. Hence we can think of the matter theory as "living" on all of AdS_2. Therefore, prior to imposing constraints, we can take the Hilbert space of the combined system to be $\mathcal{H}_0 \otimes \mathcal{H}^{\text{matt}}$, where $\mathcal{H}_0$ is defined as in section 2.2.

On this we have to impose the $\widetilde{SL}(2,\mathbb{R})$ constraints. The relevant constraint operators are now the sum of the constraint operators of the gravitational sector and the matter system:

$$J_a = J_a^R + J_a^L + J_a^{\text{matt}}. \tag{2.35}$$

Now it is straightforward to impose the constraints and construct the physical Hilbert space $\mathcal{H}$. We define $\mathcal{H}$ to be the space of coinvariants of the action of $\widetilde{SL}(2,\mathbb{R})$ on $\mathcal{H}_0 \otimes \mathcal{H}^{\rm matt}$. As before, because $\widetilde{SL}(2,\mathbb{R})$ can be used to fix $T = T' = 0$, $\chi = \chi'$ in a unique fashion, the coinvariants are generated by states of the form

$$\Psi = \delta(T)\delta(T')\delta(\chi - \chi')\psi(\chi). \tag{2.36}$$

The only difference is that $\psi(\chi)$, instead of being complex-valued, is now valued in the matter Hilbert space $\mathcal{H}^{\rm matt}$. Evaluation of the inner product (2.28) now gives

$$\langle \Psi, \Psi \rangle = \int_{-\infty}^{\infty} {\rm d}\chi \, (\psi(\chi), \psi(\chi)), \tag{2.37}$$

where here $(\ ,\ )$ is the inner product on $\mathcal{H}^{\rm matt}$. So the Hilbert space of coinvariants is $\mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{H}^{\rm matt}$, where $L^2(\mathbb{R})$ is the space of L^2 functions of χ. The algebra of operators on $\mathcal{H}$ is generated by χ, $\widetilde{p}_\chi$, and the operators on $\mathcal{H}^{\rm matt}$.

Now we want to identify the boundary Hamiltonians H_R and H_L as operators on $\mathcal{H}$. To do this, we just have to generalize eqn. (2.31) to include $J_a^{\rm matt}$. On a state of the form (2.36), the constraint operators J_a act by

$$\begin{aligned} J_1\Psi &= (p_T + p_{T'} + J_1^{\rm matt})\Psi \\ J_2\Psi &= (p_T - p_{T'} + J_2^{\rm matt})\Psi \\ J_3\Psi &= (p_\chi - p_{\chi'} + J_3^{\rm matt})\Psi. \end{aligned} \tag{2.38}$$

With the aid of these formulas, one finds that as operators on $\mathcal{H}$,

$$\begin{aligned} 2\varphi_b H_R &= \frac{1}{8}(\widetilde{p}_\chi - J_3^{\rm matt})^2 - \frac{1}{2}(J_1^{\rm matt} + J_2^{\rm matt})e^\chi + \frac{1}{2}e^{2\chi} \\ 2\varphi_b H_L &= \frac{1}{8}(\widetilde{p}_\chi + J_3^{\rm matt})^2 - \frac{1}{2}(J_1^{\rm matt} - J_2^{\rm matt})e^\chi + \frac{1}{2}e^{2\chi}. \end{aligned} \tag{2.39}$$

The operators $(\widetilde{p}_\chi \pm J_3^{\rm matt})^2$, $e^{2\chi}$, and e^χ are manifestly positive, and in a moment, we will show that the operators $-(J_1^{\rm matt} \pm J_2^{\rm matt})$ are non-negative. So H_L and H_R are positive as operators on the physical Hilbert space $\mathcal{H}$. One can also verify using eqn. (2.39) that $[H_L, H_R] = 0$, as expected since this is true even before imposing the constraints.

To understand the statement that the operators $-(J_1^{\rm matt} \pm J_2^{\rm matt})$ are non-negative, we need to discuss in more detail the meaning of the constraints. Let Φ be one of the matter fields that can be inserted on the boundary of AdS$_2$, say on the right side. The constraints are supposed to commute with boundary insertions such as $\Phi(T(t))$, while reparameterising T. Since $J_1^R = p_T = -{\rm i}\partial_T$, we have $[J_1^R, T(t)] = -{\rm i}$. To get $[J_1^R + J_1^{\rm matt}, \Phi(T(t))] = 0$, we then need $[J_1^{\rm matt}, \Phi(T)] = +{\rm i}\partial_T\Phi(T)$. Comparing to the standard quantum mechanical formula $[H, \Phi(T)] = -{\rm i}\partial_T\Phi$, where H is the Hamiltonian, we conclude that actually $J_1^{\rm matt} = -H$. In quantum field theory in AdS$_2$, H is non-negative and annihilates only the $\widetilde{SL}(2,\mathbb{R})$-invariant ground state. So therefore $J_1^{\rm matt}$ is non-positive. For $-1 < a < 1$, the operator $J_1^{\rm matt} + aJ_2^{\rm matt}$ is conjugate in $\widetilde{SL}(2,\mathbb{R})$ to a positive multiple of $J_1^{\rm matt}$, so it is again non-positive. Taking the limit $|a| \to 1$, the operators $J_1^{\rm matt} \pm J_2^{\rm matt}$ are likewise non-positive, and therefore $-(J_1^{\rm matt} \pm J_2^{\rm matt})$ is non-negative, as claimed in the last paragraph.

More generally, the operators J_a^R act on $T(t)$ by

$$[J_a^R, T] = -\mathrm{i} f_a(T), \tag{2.40}$$

where $f_a(T) = (1, \cos T, \sin T)$, and the same logic implies that

$$[J_a^{\mathrm{matt}}, \Phi(T)] = +\mathrm{i} f_a(T)\partial_T \Phi(T). \tag{2.41}$$

One might worry that the relative sign between eqn. (2.41) and eqn. (2.40) would spoil the $\widetilde{SL}(2,\mathbb{R})$ commutation relations, but actually this sign is needed for the commutation relations to work out correctly.[7]

We will describe in a little more detail the relation of boundary operators of the matter system to bulk quantum fields. Typically in the AdS/CFT correspondence, with a metric along the boundary of the local form $\frac{1}{r^2}(-\mathrm{d}T^2 + \mathrm{d}r^2)$, if a bulk field $\phi(r,T)$ vanishes for $r \to 0$ as r^Δ, then a corresponding boundary operator Φ_Δ of dimension Δ is defined by

$$\Phi_\Delta(T) = \lim_{r\to 0} r^{-\Delta}\phi(r,T). \tag{2.42}$$

In the context of JT gravity coupled to matter, we want to view both r and T as functions of the time t of the boundary quantum mechanics. Moreover, since the $\widetilde{SL}(2,\mathbb{R})$ symmetry is spontaneously broken along the boundary by the cutoff field χ, it is possible to define the boundary operator to have dimension 0, not dimension Δ. The starting point in our present derivation was the AdS_2 metric $\mathrm{d}\sigma^2 - \cosh^2\sigma\,\mathrm{d}T^2$, which for $\sigma \to \infty$ can be approximated as $\frac{1}{r^2}(-\mathrm{d}T^2 + \mathrm{d}r^2)$ with $r = 2e^{-\sigma} = \frac{\epsilon}{\varphi_b} e^{\chi}$. So $r^{-\Delta}\phi(r,T) = \left(\frac{\epsilon}{\varphi_b}\right)^{-\Delta} e^{-\Delta\chi(t)}\phi(\chi(t),T(t))$. Since $e^{-\Delta\chi(t)}$ is already one of the observables in the boundary description (before imposing constraints), we can omit this factor and define

$$\Phi(t) = \left(\frac{\epsilon}{\varphi_b}\right)^{-\Delta} \phi(\chi(t),T(t)) \tag{2.43}$$

as a boundary observable. The advantage is that $\Phi(t)$ defined this way is $\widetilde{SL}(2,\mathbb{R})$-invariant.

Before imposing constraints, it is manifest that the left Hamiltonian H_L commutes with operators inserted on the right boundary, and vice-versa. The same is therefore also true after imposing constraints. Explicitly, at $T_R = 0$,

$$[J_1^{\mathrm{matt}}, \phi(\chi(t),T(t))] = [J_2^{\mathrm{matt}}, \phi(\chi(t),T(t))] = +i\partial_T\phi(\chi(t),T(t)), \tag{2.44}$$

[7] Concretely, we have $[J_a^{\mathrm{matt}}, [J_b^{\mathrm{matt}}, \Phi(T)]] = -f_b\partial_T(f_a\partial_T\Phi)$, leading to

$$[J_a^{\mathrm{matt}}, [J_b^{\mathrm{matt}}, \Phi(T)]] - [J_b^{\mathrm{matt}}, [J_a^{\mathrm{matt}}, \Phi(T)]] = +(f_a\partial_T f_b - f_b\partial_T f_a)\partial_T\Phi(T).$$

By contrast,

$$[J_a^R, [J_b^R, T]] - [J_b^R, [J_a^R, T]] = -(f_a\partial_T f_b - f_b\partial_T f_a).$$

The commutation relations are satisfied, since the signs on the right hand sides of those two formulas are opposite, like the signs on the right hand sides of (2.40) and (2.41).

while

$$[\widetilde{p}_\chi, \phi(\chi(t), T(t))] = -[J_3^{\rm matt}, \phi(\chi(t), T(t))] = -i\Delta\, \phi(\chi(t), T(t)). \tag{2.45}$$

H_L is constructed from $\widetilde{p}_\chi + J_3^{\rm matt}$, $J_1^{\rm matt} - J_2^{\rm matt}$, and e^χ, all of which commute with $\phi(\chi(t), T(t))$. So $[H_L, \phi(\chi(t), T(t))] = 0$.

3 The Algebra

In the rest of this paper, we will study the algebra of observables in JT gravity, in general coupled to a matter theory.

In quantum field theory in a fixed spacetime M, one can associate an algebra $\mathcal{A}_\mathcal{U}$ of observables to any open set $\mathcal{U}$ in spacetime. In a theory of gravity, one has to be more careful, since spacetime is fluctuating and in general it is difficult to specify a particular region in spacetime. To the extent that fluctuations in the spacetime are small, one has an approximate notion of a spacetime region and a corresponding algebra. In JT gravity, however, at low temperatures or energies, the spacetime fluctuations are not small, so we cannot usefully define an algebra associated to a general bulk spacetime region.

Instead, as in the AdS/CFT correspondence, we can define an algebra of boundary observables. In the AdS/CFT correspondence, this would be an algebra of observables of the conformal field theory (CFT) on the boundary, possibly restricted to a region of the boundary. In favorable cases, one has some independent knowledge of the boundary CFT. In JT gravity coupled to a two-dimensional quantum field theory, there is not really a full-fledged boundary quantum mechanics, since there is no one-sided Hilbert space. But one can nevertheless define an algebra of boundary observables. More precisely, one can define algebras $\mathcal{A}_R$ and $\mathcal{A}_L$ of observables on the right and left boundaries. These will be the main objects of study in the rest of this article.

3.1 Warm up: Pure JT Gravity

Before considering theories with matter, it is helpful to first study the simpler case of pure JT gravity. As we saw in section 2.2, even in pure JT gravity, imposing the $\widetilde{SL}(2,\mathbb{R})$ constraints on the Hilbert space required working with coinvariants. At the level of operators, however, imposing the constraints simply means restricting to operators that commute with the group of constraints.

We would like to associate subalgebras $\mathcal{A}_R$ and $\mathcal{A}_L$ of gauge-invariant operators to the right and left boundaries. Classically, in JT gravity without matter, an observable on the right boundary is an $\widetilde{SL}(2,\mathbb{R})$-invariant function on the unconstrained phase space $\mathit{\Phi}_R$ of the right boundary. Here $\mathit{\Phi}_R$ is four-dimensional, and the constraint group is three-dimensional, so the quotient $\Lambda_R = \mathit{\Phi}_R/\widetilde{SL}(2,\mathbb{R})$ is one-dimensional. So classically, the algebra of $SL(2,\mathbb{R})$-invariant functions on $\mathit{\Phi}_R$ is generated by a single function that parametrizes Λ_R. For this function, we can choose the Hamiltonian H_R. Similarly, the algebra of invariant functions on the left boundary is generated by H_L. H_R and H_L are equal in classical JT gravity without matter after imposing the constraints [5, 6, 8, 11, 12].

All of these statements remain valid quantum mechanically. The only gauge-invariant right and left boundary operators are functions of the Hamiltonians H_R and H_L respectively, which are equal as operators on the constrained Hilbert space (as we saw in section 2.2). Thus in JT gravity without matter, the boundary algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ are commutative and equal and generated only by $H = H_R = H_L$. Because H has a nondegenerate spectrum, any operator that commutes with H is actually a function of H and is contained in both $\mathcal{A}_L$ and $\mathcal{A}_R$. So the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ are commutants, meaning that $\mathcal{A}_R$ is the algebra of operators that commute with $\mathcal{A}_L$, and vice-versa.

Given any algebra $\mathcal{A}$, "states" on $\mathcal{A}$ are defined to be normalized, positive linear functionals – linear maps from $\mathcal{A}$ to complex-valued "expectation values" such that positive operators have real positive expectation values and the expectation value of the identity is 1. Because the algebra $\mathcal{A}_R$ is classical, these states are in fact in one-to-one correspondence with probability distributions $p(H_R)$, where the expectation value of a function $f(H_R)$ is

$$\langle f(H_R)\rangle_p = \int_0^\infty dH_R\, p(H_R) f(H_R). \tag{3.1}$$

It is natural to ask whether one can define a notion of entropy for such states, and indeed one can. An obvious definition is the continuous (or differential) Shannon entropy

$$S(p) = -\int_0^\infty dH_R\, p(H_R)\log p(H_R). \tag{3.2}$$

There are two problems with this definition, however. The first problem is that it gives completely different answers to those given by Euclidean path integral computations. The second, related problem is that the continuous Shannon entropy is not invariant under reparameterisations where H_R is replaced by $\widetilde{H}_R = g(H_R)$ for some arbitrary invertible function g. The probability distribution $\widetilde{p}(\widetilde{H}_R)$ for $\widetilde{H}_R$ by definition satisfies

$$\langle f(H_R)\rangle_p = \int dH_R\, p(H_R) f(H_R) = \int d\widetilde{H}_R\, \widetilde{p}(\widetilde{H}_R) f(g^{-1}(\widetilde{H}_R)). \tag{3.3}$$

However, this means that the continuous Shannon entropy

$$\widetilde{S}(p) = -\int d\widetilde{H}_R\, \widetilde{p}(H_R)\log\widetilde{p}(H_R) = -\int dH_R\, p(H_R)\log\left(\left[\frac{dg}{dH_R}\right]^{-1} p(H_R)\right) \tag{3.4}$$

defined using $\widetilde{H}_R$ does not agree with the entropy (3.2) defined using H_R. In fact this second problem mildly ameliorates the first: if we choose g to be the integral of the Euclidean density of states then one obtains the "correct" Euclidean answer for the entropy. However there is nothing within the canonically quantised theory that picks out this choice of g. Without additional input from Euclidean path integral calculations, any other choice appears equally valid.

The origin of this ambiguity can be understood as follows. A state p is a linear functional on an algebra $\mathcal{A}$. However to define an entropy we need to associate to this state an operator $\rho \in \mathcal{A}$ that is normally called the density matrix of p. The state p and the density matrix ρ are related by

$$\langle \mathsf{a}\rangle_p = \mathrm{Tr}\,[\rho \mathsf{a}]. \tag{3.5}$$

for any $\mathsf{a} \in \mathcal{A}$. Here the trace Tr on the algebra $\mathcal{A}$ is some faithful positive linear functional[8] on $\mathcal{A}$ such that

$$\mathrm{Tr}[\mathsf{ab}] = \mathrm{Tr}[\mathsf{ba}] \tag{3.6}$$

for all $a, b \in \mathcal{A}$. The entropy is then defined by the usual formula

$$S(p) = -\mathrm{Tr}[\rho \log \rho] = -\langle \log \rho \rangle_p \tag{3.7}$$

However, for a commutative algebra such as $\mathcal{A}_R$, the condition (3.6) is trivial. As a result, any faithful positive linear functional is a valid choice of trace. The particular trace being used needs to be specified as part of the definition of the entropy $S(p)$. For example, if we define the trace by

$$\mathrm{Tr}[f(H_R)] = \int dH_R f(H_R), \tag{3.8}$$

then the density matrix of a state p is simply the probability distribution $\rho = p(H_R)$ viewed as an operator in $\mathcal{A}_R$. We find that $S(p)$ is the continuous Shannon entropy with respect to H_R. If (3.8) is replaced by some other positive linear functional (e.g. by replacing H_R by $\widetilde{H}_R$), then one can obtain other definitions of entropy (one for each choice of functional), including e.g. the Euclidean definition. In the absence of a preferred choice of trace included as an independent element of the theory, all of these definitions are equally natural.[9]

3.2 Definition using canonical quantisation

The fact that the boundary algebras in pure JT gravity have a nontrivial center in the intersection $\mathcal{A}_L \cap \mathcal{A}_R$ is in contrast with the general expectation in AdS/CFT that each asymptotic boundary constitutes an independent set of degrees of freedom; it has therefore been dubbed the factorisation problem [8].[10] As we shall now see, adding matter to the theory replaces the commutative boundary algebras by Type II_∞ von Neumann factors. The center is thus rendered trivial, although, because the algebras are Type II rather than Type I, the Hilbert space does not factorize into a tensor product of Hilbert spaces on each boundary, as would be expected in full AdS/CFT at finite N,

On the right boundary, we have the Hamiltonian H_R and also the QFT observables $\Phi(t)$ at an arbitrary value of the quantum mechanical time t, inserted at the corresponding point $(\chi(t), T(t))$ on the right boundary. These operators generate the right algebra $\mathcal{A}_R$. Of course, H_R generates the evolution in t:

$$\Phi(t) = e^{\mathrm{i}H_R t}\Phi(0)e^{-\mathrm{i}H_R t}. \tag{3.9}$$

[8]Here faithful means that the trace of any nonzero positive operator is nonzero. This condition is required to ensure the existence and uniqueness of ρ.

[9]From an algebraic perspective, the defect operators of [5, 12] play exactly this role; they are additional structure added to the theory that picks out a preferred choice of trace.

[10]We are using a convention here suggested by Henry Maxfield where different spellings are used to contrast this problem with the (related) factorization problem, where spacetime wormholes cause partition functions not to factorize on a set of disconnected asymptotic spacetime boundaries.

However, in order to make possible simple general statements, we want to define $\mathcal{A}_R$ as a von Neumann algebra, acting on the Hilbert space $\mathcal{H}$ that was analyzed in sections 2.2, 2.3. For this, we should consider not literally H_R and $\Phi(t)$ but bounded functions of those operators. Examples of bounded functions of H_R are $e^{\mathrm{i}H_R t}$ and (since $H_R \geq 0$) $\exp(-\beta H_R)$, with $t \in \mathbb{R}$, $\beta > 0$. For $\Phi(t)$, matters are more subtle. Experience with ordinary quantum field theory (in the absence of gravity) indicates that expressions such as $\Phi(t)$ are really operator-valued distributions, which first have to be smeared to define an operator (a densely defined unbounded operator, to be precise); then one can consider bounded functions of such operators. One can smear in real time, defining

$$\Phi_f = \int \mathrm{d}t\, f(t)\Phi(t), \tag{3.10}$$

where $f(t)$ is a smooth function of compact support, or one can smear by imaginary time evolution, defining $\Phi_\epsilon(t) = \exp(-\epsilon H_R)\Phi(t)\exp(-\epsilon H_R)$, $\epsilon > 0$.

Similarly, the left boundary $\mathcal{A}_L$ is generated by bounded functions of H_L and matter operators $\Phi_L(t)$, inserted at the position $(\chi'(t), T'(t))$ of the left boundary at quantum mechanical time t.

We would like to establish a few basic facts about these algebras:

(1) They commute with each other; more specifically the commutant of $\mathcal{A}_L$, which is defined as the algebra $\mathcal{A}_L'$ of all bounded operators on $\mathcal{H}$ that commute with $\mathcal{A}_L$, satisfies $\mathcal{A}_L' = \mathcal{A}_R$, and likewise $\mathcal{A}_R' = \mathcal{A}_L$.

(2) In the absence of matter, $\mathcal{A}_R$ and $\mathcal{A}_L$ were commutative, with the single generators $H_L = H_R$. However, after coupling to a matter QFT that satisfies reasonable conditions, we expect that $\mathcal{A}_R$ and $\mathcal{A}_L$ become "factors," meaning that their centers are trivial, and consist only of complex scalars.

(3) In the presence of matter, $\mathcal{A}_R$ and $\mathcal{A}_L$ are algebras of Type II_∞. (In the absence of matter, they are, as just noted, commutative, and therefore are direct integrals of Type I factors.)

Some of these assertions are most transparent in the context of a Euclidean-style construction of the algebras which we present in section 3.3. Here we will make some general remarks.

$\mathcal{A}_L$ is generated by left boundary operators at time zero, together with H_L. We do not need to include $\Phi(t)$ for $t \neq 0$ as an additional generator, since it is obtained from $\Phi(0)$ by conjugation by $e^{\mathrm{i}tH_L}$. Similarly, $\mathcal{A}_R$ is generated by right boundary operators at time zero together with H_R. But at time zero, the matter operators and Hamiltonian on the left boundary commute with the matter operators and the Hamiltonian on the right boundary, and vice-versa. This statement is true even before imposing constraints. So $\mathcal{A}_L$ and $\mathcal{A}_R$ commute, a statement that is conveniently written $[\mathcal{A}_L, \mathcal{A}_R] = 0$. As was already explained in section 2.2, the assertion $[\mathcal{A}_L, \mathcal{A}_R] = 0$ is a statement of causality, a quantum version of the statement that the left and right boundaries are out of causal contact.

The sharper statement $\mathcal{A}_L = \mathcal{A}_R'$, $\mathcal{A}_R = \mathcal{A}_L'$ means that the set of operators generated by $\mathcal{A}_L$ and $\mathcal{A}_R$ together is complete, in the sense that the algebra $B(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ is the same as the algebra $\mathcal{A}_L \vee \mathcal{A}_R$ generated by $\mathcal{A}_L$ and $\mathcal{A}_R$ together.

Semiclassically, one might think that this is not the case, since JT gravity coupled to matter can describe long wormholes, and one might think that operators acting deep in the interior of the long wormhole, far from the horizons of an observer on the left or right side, would not be contained in $\mathcal{A}_L \vee \mathcal{A}_R$. Entanglement wedge reconstruction, however, motivates the idea that the algebra $\mathcal{A}_L \vee \mathcal{A}_R$ is nevertheless complete, with $\mathcal{A}_L$ accounting for operators that act to the left of the RT or HRT surface, and $\mathcal{A}_R$ accounting for operators that act to the right. But entanglement wedge reconstruction is really only formulated and understood in semiclassical situations, that is, under the assumption that there is a definite semiclassical spacetime, to a good approximation. In JT gravity coupled to matter, at low temperatures or energies, that is far from being the case. Thus the statement $\mathcal{A}_L' = \mathcal{A}_R$, $\mathcal{A}_R' = \mathcal{A}_L$ can be viewed as being at least a partial counterpart of entanglement wedge reconstruction that holds even without a semiclassical picture of spacetime.

The relation to entanglement wedge reconstruction – which is a very subtle, nonclassical statement in the case that a long wormhole is present – suggests that there will be no immediate, direct argument to show that $\mathcal{A}_L' = \mathcal{A}_R$, $\mathcal{A}_R' = \mathcal{A}_L$. However, these facts will be evident in the Euclidean-style approach.

Now we discuss the question of the centers of the algebras $\mathcal{A}_R$, $\mathcal{A}_L$. For it to be true that these algebras have trivial center after coupling to a bulk QFT, it has to be the case that the QFT itself does not have any boundary operators that are central. (A non-trivial condition is needed, because abstractly we could tensor a matter QFT on AdS_2 with a topological field theory that lives only on the conformal boundary of AdS_2 and that might have central operators.) For example, we expect that there are no central boundary operators if all boundary operators $\Phi(t)$ of the QFT are limits of bulk operators $\phi(r, T)$ by the limiting procedure described in eqn. (2.43). In that case, operator products such as $\Phi(t) \cdot \Phi(t')$ will inherit short distance singularities from the singularities of bulk operator products $\phi(r_1, T_1) \cdot \phi(r_2, T_2)$, and so $\Phi(t)$ will be non-central. These short distance singularities also imply that $\Phi(t)$ depends nontrivially on t, implying after coupling to JT gravity that H_R does not commute with $\Phi(t)$ and is non-central.

Of course, one might ask whether $\mathcal{A}_R$ contains some other more complicated operator that is central. We do not have a formal proof that no such operator exists (other than c-numbers), but we find the possibility that one does highly implausible on general physical grounds. The dynamics of JT gravity are chaotic, which should mean that there are no conserved charges except for obvious ones. A central operator would be much more special than a new conserved quantity, since a conserved quantity only needs to commute with the Hamiltonian, while a central operator has to commute with every element of the algebra. A more precise argument can be made in the high-energy limit, where the fluctuations of the boundary particle become small. In that limit, the algebra $\mathcal{A}_R$ becomes the crossed product of the algebra of bulk QFT operators in the boundary causal wedge by its modular automorphism group [16, 17]. And one can prove that this crossed product algebra has trivial center whenever the bulk QFT algebra is a Type III_1 von Neumann factor. As a result, any hypothetical central operator in $\mathcal{A}_R$ would have to act trivially at high energies.

Finally we discuss the assertion that in the presence of matter, $\mathcal{A}_R$ and $\mathcal{A}_L$ are of Type II_∞. Once one knows that $\mathcal{A}_R$ or $\mathcal{A}_L$ is a factor, to assert that it is of Type II_∞ means that on this algebra one can define a trace which is positive but is not defined for all elements of the algebra.[11] Here a trace on an algebra $\mathcal{A}$ is a complex-valued linear function $\mathrm{Tr}:\mathcal{A}\to\mathbb{C}$ such that $\mathrm{Tr}\,\mathsf{a}\mathsf{a}'=\mathrm{Tr}\,\mathsf{a}'\mathsf{a}$, $\mathsf{a},\mathsf{a}'\in\mathcal{A}$; the trace is called positive if $\mathrm{Tr}\,\mathsf{a}\mathsf{a}^\dagger>0$ for all $\mathsf{a}\neq 0$.

We can argue as follows that the algebras $\mathcal{A}_R$ and $\mathcal{A}_L$ do have such a trace. For this, we consider first the thermofield double state of the two-sided system at inverse temperature β. Although JT gravity coupled to matter does not have a one-sided Hilbert space, there is a natural definition in this theory of thermal expectation values of boundary operators. For an operator $\mathsf{a}\in\mathcal{A}_R$ (or $\mathcal{A}_L$), its thermal expectation value at inverse temperature β, denoted $\langle\mathsf{a}\rangle_\beta$, is defined by evaluating a Euclidean path integral on a disc whose boundary has a renormalized length β, with an insertion of the operator a on the boundary. Alternatively, there is a thermofield double state $\Psi_{\rm TFD}(\beta)$ such that thermal expectation values are equal to expectation values in the thermofield double state:

$$\langle\Psi_{\rm TFD}(\beta)|\mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle=\langle\mathsf{a}\rangle_\beta. \tag{3.11}$$

In the case of JT gravity with or without matter, the thermofield double description is not obtained by doubling anything, since there is no one-sided Hilbert space. However, in the two-sided Hilbert space of JT gravity, there is a state $\Psi_{\rm TFD}(\beta)$ that satisfies eqn. (3.11) [5, 6, 8, 34]. It can be defined by a path integral on a half-disc with an asymptotic boundary of renormalized length $\beta/2$ (and a geodesic boundary on which the state is defined; see section 3.3). Defined this way, $\Psi_{\rm TFD}(\beta)$ is not in general normalized, but satisfies $\langle\Psi_{\rm TFD}(\beta)|\Psi_{\rm TFD}(\beta)\rangle=Z(\beta)$ where $Z(\beta)$ is the Euclidean partition function on a disc with renormalized boundary length β. Because this Euclidean path integral has no matter operator insertions, any matter fields present are in the $\widetilde{SL}(2,\mathbb{R})$-invariant ground state $\Psi_{\rm gs}$. Therefore, the thermofield double state in the presence of matter is simply the tensor product of the thermofield double state $\Psi_{\rm TFD}(\beta)$ for pure JT gravity with $\Psi_{\rm gs}\in\mathcal{H}^{\rm matt}$.

As in the case of an ordinary quantum system, expectation values in the thermofield double state satisfy a KMS condition:

$$\langle\Psi_{\rm TFD}(\beta)|\Phi(t)\Phi(t')|\Psi_{\rm TFD}(\beta)\rangle=\langle\Psi_{\rm TFD}(\beta)|\Phi(t')\Phi(t+\mathrm{i}\beta)|\Psi_{\rm TFD}(\beta)\rangle. \tag{3.12}$$

More generally, for any $\mathsf{a},\mathsf{a}'\in\mathcal{A}_R$, with the definition $\mathsf{a}(t)=e^{\mathrm{i}H_Rt}\mathsf{a}e^{-\mathrm{i}H_Rt}$, we have

$$\langle\Psi_{\rm TFD}(\beta)|\mathsf{a}\mathsf{a}'|\Psi_{\rm TFD}(\beta)\rangle=\langle\Psi_{\rm TFD}|\mathsf{a}'\mathsf{a}(\mathrm{i}\beta)|\Psi_{\rm TFD}(\beta)\rangle. \tag{3.13}$$

We define $\mathrm{Tr}\,\mathsf{a}=\lim_{\beta\to0}\langle\Psi_{\rm TFD}(\beta)|\mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle$ for any $\mathsf{a}\in\mathcal{A}_R$ such that this limit exists. The limit certainly does not exist for all a; for example, if $\mathsf{a}=1$, then $\langle\Psi_{\rm TFD}(\beta)|\mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle$ is equal to the partition function $Z(\beta)$, which diverges for $\beta\to0$. But it is equally clear that there exist $\mathsf{a}\in\mathcal{A}_R$ such that the limit does exist. For example, for $\mathsf{a}=e^{-\epsilon H_R}$, $\epsilon>0$, we get $\lim_{\beta\to0}\langle\Psi_{\rm TFD}(\beta)|\mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle=\lim_{\beta\to0}Z(\beta+\epsilon)=Z(\epsilon)$, so a (and similarly any operator regularized by a factor such as $\exp(-\epsilon H_R)$) has a well-defined trace. For operators

[11] $\mathcal{A}_R$ and $\mathcal{A}_L$ are not of Type I, since in JT gravity coupled to matter, there is no one-sided Hilbert space.

such that the limits exist, the $\beta \to 0$ limit of the KMS condition shows that the function Tr satisfies the defining property of a trace. As for positivity, one has

$$\langle \Psi_{\rm TFD}(\beta)|\mathsf{a}^\dagger \mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle = \langle \mathsf{a}\Psi_{\rm TFD}(\beta)|\mathsf{a}\Psi_{\rm TFD}(\beta)\rangle \geq 0, \tag{3.14}$$

with vanishing if and only if $\mathsf{a}\Psi_{\rm TFD}(\beta) = 0$. Since H_L commutes with $\mathsf{a}, \mathsf{a}^\dagger \in \mathcal{A}_R$, and $e^{-(\beta_2-\beta_1)H_L/2}\Psi_{\rm TFD}(\beta_1) = \Psi_{\rm TFD}(\beta_2)$, we have

$$\begin{aligned}\langle \Psi_{\rm TFD}(\beta_2)|\mathsf{a}^\dagger \mathsf{a}|\Psi_{\rm TFD}(\beta_2)\rangle &= \langle \Psi_{\rm TFD}(\beta_1)|\mathsf{a}^\dagger \exp(-(\beta_2-\beta_1)H_L)\mathsf{a}|\Psi_{\rm TFD}(\beta_1)\rangle \\ &\leq \langle \Psi_{\rm TFD}(\beta_1)|\mathsf{a}^\dagger \mathsf{a}|\Psi_{\rm TFD}(\beta_1)\rangle \end{aligned} \tag{3.15}$$

for $\beta_2 > \beta_1$. Hence (3.14) is a monotonically decreasing function of β. Thus as $\beta \to 0$, (3.14) always either converges to a finite positive limit or tends to positive infinity. We conclude that $\mathrm{Tr}(\mathsf{a}^\dagger\mathsf{a}) \in [0, +\infty]$ is in fact well defined in the extended positive real numbers for any positive operator $\mathsf{a}^\dagger\mathsf{a}$. We will argue in section 3.3 that the algebras $\mathcal{A}_R, \mathcal{A}_L$ are cyclic-separating for $\Psi_{\rm TFD}(\beta)$. As a result, $\mathsf{a}\Psi_{\rm TFD}(\beta) = 0$ implies $\mathsf{a} = 0$ and the trace Tr is faithful. We should add that the existence of a faithful trace will anyway be perhaps more obvious in section 3.3.

There is an alternative definition of the trace Tr that was used in Appendix I of [33] to give an algorithm for computing Euclidean disc partition functions from canonically quantised pure JT gravity (although the interpretation as an algebraic trace on the boundary algebras was not noted there). In the high temperature limit, the wavefunction $\Psi_{\rm TFD}(\beta)$ becomes tightly peaked as a function of χ around a saddle-point value χ_c such that $\chi_c \to \infty$ as $\beta \to 0$. Equivalently, it is peaked around a semiclassical renormalized geodesic length $\ell_c = -2\chi_c$ such that $\ell_c \to -\infty$ as $\beta \to 0$. As a result the trace of an operator a with matrix elements $\mathsf{a}(\chi_1, \chi_2) \in \mathcal{B}(\mathcal{H}^{\rm matt})$ is given by

$$\mathrm{Tr}(\mathsf{a}) = \lim_{\chi\to\infty} \exp(\chi + 8e^{\chi})\langle \Psi_{\rm gs}|\mathsf{a}(\chi,\chi)|\Psi_{\rm gs}\rangle. \tag{3.16}$$

The correct scaling of the prefactor in (3.16) may be determined from the normalization of $\Psi_{\rm TFD}(\beta)$ as a function of the saddle-point value χ_c as $\beta \to 0$. Alternatively, it may be determined by analyzing the universal decay as $\chi \to \infty$ of the matrix elements of operators that e.g. project onto finite-energy states and hence should have finite trace.

Let us use this trace to compute the entanglement entropy of the thermofield double state $\Psi_{\rm TFD}(\beta)$, or, more precisely, of the normalized thermofield double state

$$\widehat{\Psi}_{\rm TFD}(\beta) = \frac{\Psi_{\rm TFD}(\beta)}{Z(\beta)^{1/2}}. \tag{3.17}$$

It follows from the definition using path integrals (and can be verified explicitly using the formulas from [6]) that

$$e^{-\beta_1 H_R/2}\Psi_{\rm TFD}(\beta_2) = \Psi_{\rm TFD}(\beta_1+\beta_2). \tag{3.18}$$

As a result, for any $\mathsf{a} \in \mathcal{A}_R$, we have

$$\langle \Psi_{\rm TFD}(\beta)|\mathsf{a}|\Psi_{\rm TFD}(\beta)\rangle = \lim_{\beta'\to 0}\langle \Psi_{\rm TFD}(\beta')|e^{-\beta H_R/2}\mathsf{a}e^{-\beta H_R/2}|\Psi_{\rm TFD}(\beta')\rangle \tag{3.19}$$

$$= \mathrm{Tr}[e^{-\beta H_R/2}\mathsf{a}e^{-\beta H_R/2}] = \mathrm{Tr}[e^{-\beta H_R}\mathsf{a}]. \tag{3.20}$$

We therefore conclude that the density matrix of the normalized thermofield double state $\widehat{\Psi}_{\text{TFD}}(\beta)$ on $\mathcal{A}_R$ is $\rho = e^{-\beta H_R}/Z(\beta)$. The entropy of this state is

$$S(\rho) = -\langle \log \rho \rangle = \langle \beta H_R \rangle + \log Z(\beta), \tag{3.21}$$

which matches the Euclidean answer.

Crucially, unlike in JT gravity without matter, we did not need to add any additional ingredients by hand in order to obtain this result: if the algebra $\mathcal{A}_R$ is a von Neumann factor, that is, its center is trivial, then the trace (if it exists) is unique up to rescaling.[12] Consequently, the entropy formula derived here is unique up to an additive constant. Even though we used Euclidean path integrals as a convenient way of discovering the trace, the definition itself was forced upon us by the structure of the algebra.

Since the algebra is Type II_∞, there is no canonical choice of normalization for the trace, and hence no canonical choice for the additive constant in the definition of entropy. There is a similar additive ambiguity in Euclidean path integral entropy computations. The JT gravity action includes a topological term that evaluates to $-S_0\chi$ where χ is the Euler characteristic of the spacetime manifold. To remove contributions from higher genus spacetimes containing wormholes, one needs to take the limit $S_0 \to \infty$. This leads to a state-independent infinite contribution S_0 to the entanglement entropy, which describes the universal divergent entanglement of the Type II_∞ algebra. To define a finite renormalized entanglement entropy we need to subtract this piece, which leads to the same additive ambiguity that we found above from an algebraic perspective.

3.3 Definition using Euclidean path integrals

We now offer an alternative definition of the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ based on Euclidean path integrals. Although we will eventually argue that this definition is equivalent to the one given above, it is helpful because a) it makes certain expected properties of $\mathcal{A}_R$ and $\mathcal{A}_L$ (such as the fact that they are commutants) easier to justify, and b) it justifies the use of Euclidean replica trick computations for computing entropies on $\mathcal{A}_R$ or $\mathcal{A}_L$.

Our starting point is a formal algebra $\mathcal{A}_0$, built out of strings of symbols, each of which is either $e^{-\beta H}$, with some $\beta > 0$, or else one of the boundary operators Φ_α of the matter system. The two types of symbol are required to alternate and the string is required to begin and end with a symbol of the type $e^{-\beta H}$. Thus here are some examples of allowed strings:

$$\begin{aligned}
&e^{-\beta H} \\
&e^{-\beta H}\Phi e^{-\beta' H} \\
&e^{-\beta_1 H}\Phi_1 e^{-\beta_2 H}\Phi_2 e^{-\beta_3 H}.
\end{aligned} \tag{3.22}$$

Strings are multiplied in an obvious way by joining them end to end and using the relation $e^{-\beta H}e^{-\beta' H} = e^{-(\beta+\beta')H}$. Thus for example if $\mathsf{S}_1 = e^{-\beta_1 H}\Phi_1 e^{-\beta_2 H}$ and $\mathsf{S}_2 = e^{-\beta_3 H}\Phi_2 e^{-\beta_4 H}$,

[12] More precisely, on a Type I or II factor, the trace is unique if one requires it to be normal and semifinite; see the discussion at the end of section 4 for details.

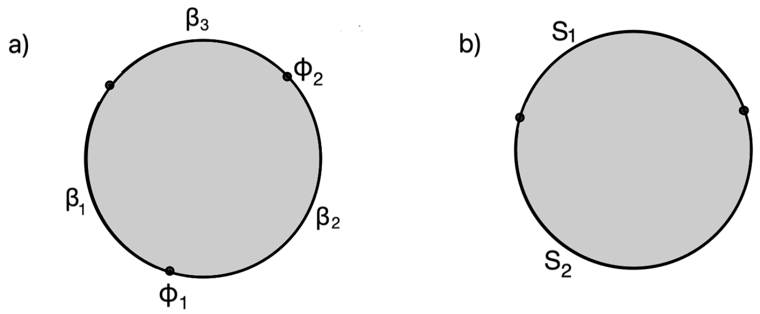

Figure 1. (a) The path integral on a disc that computes $\mathrm{Tr}\,\mathsf{S}$ with $\mathsf{S} = e^{-\beta_1 H}\Phi_1 e^{-\beta_2 H}\Phi_2 e^{-\beta_3 H}$. The boundary of the disc is made of three segments with renormalized lengths β_1, β_2, and β_3. At two junctions of segments, operators Φ_1 and Φ_2 are inserted. At the third junction, the two ends of S are joined together. (b) The path integral on a disc that computes $\mathrm{Tr}\,\mathsf{S}_1\mathsf{S}_2$. The boundary of the disc consists of two segments labeled respectively by S_1 and by S_2. There is no intrinsic ordering of the two segments so $\mathrm{Tr}\,\mathsf{S}_1\mathsf{S}_2 = \mathrm{Tr}\,\mathsf{S}_2\mathsf{S}_1$.

then $\mathsf{S}_1\mathsf{S}_2 = e^{-\beta_1 H}\Phi_1 e^{-(\beta_2+\beta_3)H}\Phi_2 e^{-\beta_4 H}$. Eventually, we will reinterpret these strings as the Hilbert space operators that these expressions usually represent, but to begin with we consider them as formal symbols.

We can define an algebra $\mathcal{A}_0$ whose elements are complex linear combinations of strings, multiplied as just explained. This is an algebra without an identity element; we could add an identity element as an additional generator of $\mathcal{A}_0$ but this will not be convenient.

The Euclidean path integral on a disc can be used to define a trace on the algebra $\mathcal{A}_0$. In this article, a disc path integral, when not otherwise specified, is a path integral on a disc whose boundary is an asymptotic boundary on which the boundary quantum mechanics is defined. Thus, in the limit that the usual cutoff is removed, the boundary of the disc is at conformal infinity in AdS_2. We do not assume time-reversal symmetry, so discs, and more general two-dimensional spacetimes considered later, are oriented, as are their boundaries. In the figures, the orientation runs counterclockwise along the boundary (thus, upwards or "forwards in imaginary time" on right boundaries and downwards or "backwards in imaginary time" on left boundaries).

To define $\mathrm{Tr}\,\mathsf{S}$ for a string S, we view S, with its ends sewn together, as a recipe to define a boundary condition on the boundary of the disc. For example (fig. 1(a)), for the case $\mathsf{S} = e^{-\beta_1 H}\Phi_1 e^{-\beta_2 H}\Phi_2 e^{-\beta_3 H}$, $\mathrm{Tr}\,S$ is computed by a path integral on a disc whose renormalized circumference is $\beta = \beta_1 + \beta_2 + \beta_3$, with insertions of the operators Φ_1 and Φ_2 at boundary points separated by imaginary time β_2. With this recipe, a simple rotation of the path integral picture shows that for any two strings S_1, S_2, we have $\mathrm{Tr}\,\mathsf{S}_1\mathsf{S}_2 = \mathrm{Tr}\,\mathsf{S}_2\mathsf{S}_1$ (fig. 1(b)). Hence Tr is indeed a trace.

So far the elements of $\mathcal{A}_0$ are just symbols, However, we can extract more information from the path integral on a disc. First, we define the "adjoint" $\mathsf{S}^\dagger$ of a string S. $\mathsf{S}^\dagger$ is defined by reversing the order of the symbols in S and replacing each matter operator Φ with its adjoint $\Phi^\dagger$. For example, the adjoint of $\mathsf{S} = e^{-\beta_1 H}\Phi e^{-\beta_2 H}$ is $\mathsf{S}^\dagger = e^{-\beta_2 H}\Phi^\dagger e^{-\beta_1 H}$. So we

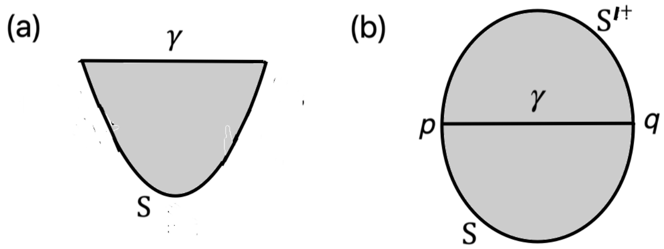

Figure 2. (a) The path integral on a half-disc that computes the map from a string S to a Hilbert space state Ψ_{S}. The half-disc has an asymptotic boundary labeled by the string S and a geodesic boundary γ. (b) The path integral that computes $\langle \mathsf{S}', \mathsf{S} \rangle$ and can be used to demonstrate that the map $\mathsf{S} \to \Psi_{\mathsf{S}}$ from a string to a bulk state preserves inner products.

can define a hermitian inner product on $\mathcal{A}_0$ by $\langle \mathsf{S}_1, \mathsf{S}_2 \rangle = \mathrm{Tr}\, S_1^\dagger S_2$. We will see shortly that this inner product is positive semi-definite but has plenty of null vectors. If $\mathcal{N}$ is the subspace of null vectors, then $\mathcal{A}_0/\mathcal{N}$ is a vector space with a positive-definite hermitian inner product. It can therefore be completed to a Hilbert space.

But in fact, this Hilbert space is none other than the Hilbert space $\mathcal{H}$ of JT gravity plus matter, described in section 2.3. We recall that an element of $\mathcal{H}$ is a square-integrable function $\Psi(\chi)$ that is valued in the matter Hilbert space $\mathcal{H}^{\mathrm{matt}}$, where the renormalized length of a geodesic between the two boundaries is $\ell = -2\chi$; in other words, $\mathcal{H} = \mathcal{H}^{\mathrm{matt}} \otimes L^2(\mathbb{R})$, where χ acts on $L^2(\mathbb{R})$ by multiplication. A path integral on what we will call a half-disc gives a linear map $\mathsf{S} \to \Psi_{\mathsf{S}} \in \mathcal{H}$. By a half-disc, we mean a disc whose boundary consists of two connected components, one an asymptotic boundary on which the dual quantum mechanics is defined, and one an "interior" boundary at a finite distance. The structure of an asymptotic boundary is defined by a string. Interior boundaries are always assumed to be geodesics. With this understanding, the path integral on a half-disc can be used to define a linear map $\mathsf{S} \to \Psi_{\mathsf{S}} \in \mathcal{H}$ (fig 2(a)). We compute Ψ_{S} by a path integral on a half-disc that has an asymptotic boundary determined by S and an interior geodesic boundary of renormalized length $\ell = -2\chi$. For given χ, the output of this path integral is a state in $\mathcal{H}^{\mathrm{matt}}$, and letting χ vary we get the desired state $\Psi_{\mathsf{S}}(\chi) \in \mathcal{H}$.

The map $\mathsf{S} \to \Psi_{\mathsf{S}}$ preserves inner products in the sense that

$$\langle \mathsf{S}', \mathsf{S} \rangle = \langle \Psi_{\mathsf{S}'}, \Psi_{\mathsf{S}} \rangle, \tag{3.23}$$

where the inner product on the left is the one on $\mathcal{A}_0$, and the inner product on the right is the one on $\mathcal{H}$. To justify eqn. (3.23), we simply consider (fig. 2(b)) the path integral that computes $\langle \mathsf{S}', \mathsf{S} \rangle = \mathrm{Tr}\, \mathsf{S}'^\dagger \mathsf{S}$. This is a path integral on a disc D with an asymptotic boundary that consists of segments labeled respectively by S and by $\mathsf{S}'^\dagger$, joined at their common endpoints p, q. In the standard procedure to analyze the path integral of JT gravity, possibly coupled to matter, the first step is to integrate over the dilaton field. This gives a delta function such that the metric on the disc becomes the standard AdS_2 metric of constant negative curvature (cut off near the conformal boundary, as reviewed in section

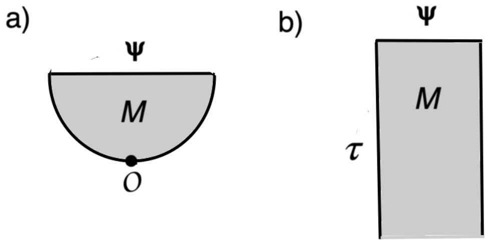

Figure 3. Two views of a spacetime M which is half of AdS_2 (in Euclidean signature). Viewing AdS_2 as a hyperbolic disc, half of AdS_2 is the half-disc shown in (a); on the other hand, the AdS_2 metric can be put in the static form $\mathrm{d}\sigma^2 + \cosh^2\sigma \mathrm{d}\tau^2$, and in this form, half of AdS_2 looks like a semi-infinite strip with $\tau \leq 0$, as shown in (b).

2.1). In this metric, there is a unique geodesic γ from p to q. This geodesic divides D into a "lower" part D_- and an "upper" part D_+. The path integral on D_- computes the ket $|\Psi_{\mathsf{S}}\rangle$, the path integral on D_+ computes the bra $\langle\Psi_{\mathsf{S}'}|$, and the integral over degrees of freedom on γ sews these two states together and computes their inner product $\langle\Psi_{\mathsf{S}'}, \Psi_{\mathsf{S}}\rangle$. So this establishes eqn. (3.23), which in particular confirms that the inner product $\langle\ ,\ \rangle$ on $\mathcal{A}_0$ is positive semi-definite,

As an example of this construction, let $\mathsf{S} = e^{-\beta H/2}$. The corresponding state $\Psi_{\mathsf{S}} = |e^{-\beta H/2}\rangle$ is actually the thermofield double state of the two-sided system, at inverse temperature β. Indeed, for this choice of S, the recipe to compute Ψ_{S} is just the standard recipe to construct the thermofield double state by a path integral on a half-disc. The thermofield double state was already discussed in section 3.2.

The map $\mathcal{A}_0 \to \mathcal{H}$ is surjective, in the sense that states of the form Ψ_{S}, $\mathsf{S} \in \mathcal{A}_0$ suffice to generate $\mathcal{H}$. This is particularly clear if the matter theory is a conformal field theory (CFT). Let Ω be the CFT ground state. The operator-state correspondence says that any state in $\mathcal{H}^{\mathrm{matt}}$ is of the form $\Phi|\Omega\rangle$ for some unique local CFT operator Φ. A consequence is that states Ψ_{S} for S of the highly restricted form $\mathsf{S} = e^{-\beta H/2}\Phi e^{-\beta H/2}$ actually suffice to generate $\mathcal{H}$. Indeed, we can choose Φ to generate any desired state of the matter system, multiplied by a function of χ that depends on β.[13] Taking linear combinations of the states we get for different values of β, we can approximate any desired function of χ; consequently, states Ψ_{S} for S of this restricted form suffice to generate $\mathcal{H}$. All of the other strings that we could have used, with more than one CFT operator, are therefore redundant in the sense that they do not enable us to produce any new states in $\mathcal{H}$. So the map $\mathcal{A}_0 \to \mathcal{H}$ has a very large space $\mathcal{N}$ of null vectors, as asserted earlier.

Even if the matter system is not conformally invariant, the same idea applies, basically because the $\widetilde{SL}(2,\mathbb{R})$ symmetry of AdS_2 is the conformal group of the boundary. The

[13]One has to be slightly careful here because the operators $e^{-\beta H/2}$ act nontrivially on the matter Hilbert space. As a result, the reduced state of Ψ_{S} on $\mathcal{H}^{\mathrm{matt}}$ will not necessarily be the state dual to Φ. However we do not expect this fact to alter the basic conclusion that a dense set of states in $\mathcal{H}$ can be prepared using strings S of the form described above.

relevant facts are actually familiar in the AdS/CFT correspondence, where typically the bulk theory is not at all conformally invariant but the boundary theory is conformally invariant, and any bulk state can be created by a local operator on the boundary. In our context, this reasoning applies to the matter sector, which possesses unbroken $\widetilde{SL}(2,\mathbb{R})$ symmetry (not to the full system including JT gravity). The basic setup is depicted in fig. 3, which shows two views of a spacetime M that is half of Euclidean AdS_2. For any matter QFT, the path integral in in (a) gives a map from a local operator O inserted at on the conformal boundary, as shown, to a bulk state Ψ observed on the upper, geodesic boundary of M. From (b), we can get a map in the opposite direction. Suppose that the state Ψ is an energy eigenstate with energy E_0. Cut off the strip by restricting to the range $-\tau_0 \leq \tau \leq 0$ and input the state Ψ at the bottom of the strip. The path integral in the strip will then give back the same state Ψ at the top, multiplied by $e^{-\tau_0 E_0}$. To compensate for this, multiply the path integral in the strip by $e^{+\tau_0 E_0}$. Then upon taking the limit $\tau_0 \to \infty$, the picture in (b) becomes equivalent to the one in (a), with a state inserted in the far past turning into a local operator O inserted on the boundary.

Now we want to show that the quotient of $\mathcal{A}_0$ by its subspace of null vectors, namely $\mathcal{A}_1 = \mathcal{A}_0/\mathcal{N}$, is an algebra in its own right and has a trace. To show that the linear function $\mathrm{Tr} : \mathcal{A}_0 \to \mathbb{C}$ makes sense as a function on $\mathcal{A}_1$, one needs to show that for $\mathsf{S} \in \mathcal{A}_0$, $\mathrm{Tr}\,\mathsf{S}$ is invariant under $\mathsf{S} \to \mathsf{S} + \mathsf{S}_0$ with $\mathsf{S}_0 \in \mathcal{N}$. In other words, one has to show that $\mathrm{Tr}\,\mathsf{S}_0 = 0$. S_0 being null means $(\mathsf{S}_1, \mathsf{S}_0) = 0$ for any S_1. In particular, taking $\mathsf{S}_1 = e^{-\epsilon H}$, we have $0 = \langle e^{-\epsilon H}, \mathsf{S}_0\rangle = \mathrm{Tr}\, e^{-\epsilon H}\mathsf{S}_0$, and hence

$$0 = \lim_{\epsilon \to 0} \mathrm{Tr}\, e^{-\epsilon H}\mathsf{S}_0 = \mathrm{Tr}\,\mathsf{S}_0, \tag{3.24}$$

as desired.

What is involved in showing that $\mathcal{A}_1 = \mathcal{A}_0/\mathcal{N}$ is an algebra in its own right? Consider two equivalence classes in $\mathcal{A}_0/\mathcal{N}$ that can be represented by elements $\mathsf{S}_1, \mathsf{S}_2 \in \mathcal{A}_0$. To be able to consistently multiply equivalence classes, we need the condition that if we shift S_1 or S_2 in its equivalence class by $\mathsf{S}_1 \to \mathsf{S}_1 + \mathsf{S}_0$ or $\mathsf{S}_2 \to \mathsf{S}_2 + \mathsf{S}_0$ where S_0 is null, then $\mathsf{S}_1\mathsf{S}_2$ should shift by a null vector. In other words, the condition we need is that if S_0 is null, then $\mathsf{S}\mathsf{S}_0$ and $\mathsf{S}_0\mathsf{S}$ are null, for any $\mathsf{S} \in \mathcal{A}_0$.

To prove this, we consider the path integral on a half-disc D_0 that computes $\Psi_{\mathsf{S}_0\mathsf{S}}$. We want to show that if S_0 is null, this path integral is identically zero, regardless of S and regardless of the renormalized length of the geodesic boundary of D_0. The boundary of D_0 consists of a geodesic, say with endpoints p and q, and an asymptotic boundary that is the union of two intervals labeled by S_0 and by S, which meet at a common endpoint r (fig. 4). Let pr be the segment labeled by S_0. The points p and r are joined in D_0 by a unique geodesic γ. This geodesic divides D_0 into two pieces. One piece is a smaller half-disc D_1 whose asymptotic boundary is labeled by S_0, and which has γ for its geodesic boundary. Let D_2 be the rest of D_0. The path integral on D_0 can be evaluated by first evaluating separately the path integrals on D_1 and on D_2, keeping fixed the fields on γ (χ and the matter fields), and then at the end integrating over the fields on γ. The statement that S_0 is null means that the path integral on D_1 vanishes, for any values of the fields on γ.

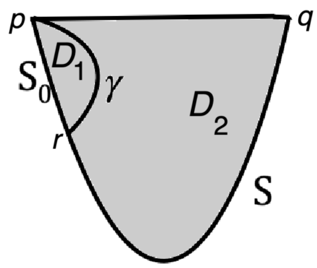

Figure 4. Depicted here is a half-disc D_0 with an asymptotic boundary labeled by $\mathsf{S}_0\mathsf{S}$ and a geodesic boundary (the horizontal line at the top). The path integral on D_0 computes $\Psi_{\mathsf{S}_0\mathsf{S}}$. γ is a geodesic that connects the endpoints p, r of the boundary segment labeled by S_0. If $\Psi_{\mathsf{S}_0} = 0$, then the path integral in the region D_1 bounded by S_0 and γ vanishes, regardless of the fields on γ, and therefore $\Psi_{\mathsf{S}_0\mathsf{S}} = 0$.

Hence the path integral on D_0 vanishes, showing that $\mathsf{S}_0\mathsf{S}$ is null. By similar reasoning, $\mathsf{S}\mathsf{S}_0$ is null if S_0 is null. Arguments similar to the one just explained will recur at several points in this article.

The function $\mathrm{Tr} : \mathcal{A}_1 \to \mathbb{C}$ obeys the usual condition $\mathrm{Tr}\,\mathsf{S}_1\mathsf{S}_2 = \mathrm{Tr}\,\mathsf{S}_2\mathsf{S}_1$, since this was already true on $\mathcal{A}_0$. Moreover, Tr is positive as a function on $\mathcal{A}_1$, in the sense that $\mathrm{Tr}\,\mathsf{S}^\dagger\mathsf{S} > 0$ for all $\mathsf{S} \neq 0$, since we have disposed of null vectors in passing to $\mathcal{A}_1$.

We can now reinterpret strings as Hilbert space operators. If S, T are strings, we say that S acts on the state Ψ_T by $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}}$. This definition is consistent, since if T is null (so that $\Psi_\mathsf{T} = 0$), then ST is also null (so $\Psi_{\mathsf{ST}} = 0$). Since states Ψ_T are dense in $\mathcal{H}$ and the operators S defined this way are bounded, the rule $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}}$ completely defines S as an operator on $\mathcal{H}$. Finally, since $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}} = 0$ if S is null, the operator corresponding to S only depends on the equivalence class of S in $\mathcal{A}_1 = \mathcal{A}_0/\mathcal{N}$. Thus we get an action of $\mathcal{A}_1$ on the Hilbert space $\mathcal{H}$.

The operator that acts on $\mathcal{H}$ by $\Psi_\mathsf{T} \to \mathsf{S}\Psi_\mathsf{T}$ is actually the standard Hilbert space operator that one would associate to the string S, acting on the left boundary of a two-sided spacetime. That is true because the path integral rules that we have given agree with the standard recipe to interpret a string S as a Hilbert space operator. To define S as an operator between states in $\mathcal{H}$, we would consider according to the standard logic a path integral on a hyperbolic two-manifold with geodesic boundaries on which initial and final states in $\mathcal{H}$ are inserted, and an asymptotic boundary labeled by S (fig. 5(a)). This path integral will compute a matrix element of S between initial and final states in $\mathcal{H}$. Now if we want to let S act on Ψ_T, we just glue onto the lower geodesic boundary in fig 5(a) the path integral construction of the state Ψ_T, adapted from fig. 2(a). The resulting picture (5(b)) is just the natural path integral construction of the state Ψ_{ST}. So the rule $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}}$ agrees with the standard definition of a Hilbert space operator corresponding to S, acting on the left boundary of a two-sided state. To get operators acting on the right boundary, we would consider the operation $\Psi_\mathsf{T} \to \Psi_{\mathsf{TS}}$. This gives the commutant or opposite algebra, as we discuss presently.

At this stage, in particular we know that $\mathcal{A}_1 = \mathcal{A}_0/\mathcal{N}$ is an algebra that acts on a

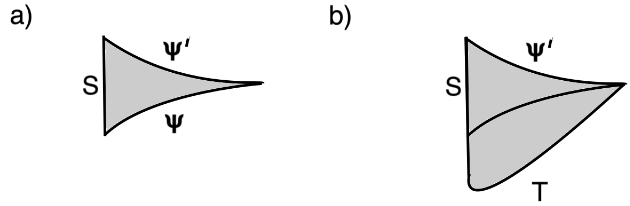

Figure 5. (a) This figure shows the path integral that would be used to compute a matrix element $\langle\Psi'|\mathsf{S}|\Psi\rangle$ of a Hilbert space operator corresponding to a string S between initial and final states Ψ, Ψ' in the bulk Hilbert space $\mathcal{H}$. Ψ and Ψ' are inserted on geodesic boundaries that asymptotically meet at a point on the right boundary. (b) In the special case that the initial state is $\Psi = \Psi_{\mathsf{T}}$, by gluing onto (a) the path integral preparation of the state Ψ_{T}, we get a representation of the matrix element $\langle\Psi'|\mathsf{S}|\Psi_{\mathsf{T}}\rangle$. But this coincides with the path integral representation we would use for the inner product $\langle\Psi'|\Psi_{\mathsf{ST}}\rangle$, showing that the standard interpretation of S as a Hilbert space operator is consistent with $\mathsf{S}\Psi_{\mathsf{T}} = \Psi_{\mathsf{ST}}$. Note that this picture can also be read to show that if $\Psi_{\mathsf{T}} = 0$ then $\Psi_{\mathsf{ST}} = 0$.

Hilbert space $\mathcal{H}$. We can therefore complete $\mathcal{A}_1$ to get a von Neumann algebra $\mathcal{A}$ that acts on $\mathcal{H}$. Although $\mathcal{A}_1$ does not contain an identity element, $\mathcal{A}$ does. The reason for this is the following. Although $\mathcal{A}_1$ does not contain an identity element, it does contain the elements $e^{-\epsilon H}$ for arbitrary $\epsilon > 0$. When we complete $\mathcal{A}_1$ to get a von Neumann algebra, we have to include all operators on $\mathcal{H}$ that occur as limits of operators in $\mathcal{A}_1$. In particular, we have to include the identity operator $\mathbf{1}$, since it arises as $\lim_{\epsilon\to 0} e^{-\epsilon H}$. The reason that we did not include an identity operator in $\mathcal{A}_0$ at the beginning is that this would have prevented us from being able to define the map $\mathsf{S} \to \Psi_{\mathsf{S}}$, since there is no Hilbert space state that corresponds to the identity operator $\mathbf{1}$. Since the state that corresponds to $e^{-\beta H/2}$ is the thermofield double state at inverse temperature β, a state $|\mathbf{1}\rangle$ corresponding to $\mathbf{1} = \lim_{\beta\to 0} e^{-\beta H/2}$ would be the infinite temperature limit of the thermofield double state. But there is no such Hilbert space state; its norm would be $\langle\mathbf{1},\mathbf{1}\rangle = \lim_{\beta\to 0} \mathrm{Tr}\, e^{-\beta H} = \lim_{\beta\to 0} Z(\beta) = \infty$. Rather, one can interpret $|\mathbf{1}\rangle$ as a "weight" of the von Neumann algebra $\mathcal{A}$, which means roughly that it is an unnormalizable state that has well-defined inner products with a dense set of elements of $\mathcal{A}$. Indeed, $\langle\mathbf{1},\mathsf{S}\rangle = \mathrm{Tr}\,\mathsf{S}$ is well-defined for any $\mathsf{S} \in \mathcal{A}_1$, and by definition $\mathcal{A}_1$ is dense in $\mathcal{A}$.

Since $\mathcal{A}_1$ has a trace that is positive-definite, the same is true of its completion $\mathcal{A}$. However, taking the completion adds to $\mathcal{A}_1$ elements – such as the identity element $\mathbf{1}$ – with trace $+\infty$. Since the trace in $\mathcal{A}$ is accordingly not defined for all elements of $\mathcal{A}$, it follows that $\mathcal{A}$ is of Type II_∞, not Type II_1. $\mathcal{A}$ is not of Type I because there is no one-sided Hilbert space for it to act on. It is not of Type III because it has a trace.

Now we can analyze the commutant $\mathcal{A}'$ of the algebra $\mathcal{A}$. What makes this straightforward is the close relation between $\mathcal{A}$ and $\mathcal{H}$: they were both obtained by completing $\mathcal{A}_1$, albeit in slightly different ways. Let T be a linear operator on $\mathcal{H}$ that commutes with

$\mathcal{A}$. Consider any $\mathsf{S}, \mathsf{U} \in \mathcal{A}_1 \subset \mathcal{A}$. For T to commute with S as operators on $\mathcal{H}$ implies in particular that $\mathsf{S}\mathsf{T}\Psi_\mathsf{U} = \mathsf{T}\mathsf{S}\Psi_\mathsf{U} = \mathsf{T}\Psi_{\mathsf{S}\mathsf{U}}$. Now set $\mathsf{U} = e^{-\epsilon H}$ and take the limit $\epsilon \to 0$. In this limit, $\mathsf{S}\mathsf{U} \to \mathsf{S}$ and $\Psi_\mathsf{U} \to |\mathbf{1}\rangle$, so we get $\mathsf{T}\Psi_\mathsf{S} = \mathsf{S}\mathsf{T}|\mathbf{1}\rangle$. We can approximate $\mathsf{T}|\mathbf{1}\rangle$ arbitrarily well by Ψ_W for some $\mathsf{W} \in \mathcal{A}_1$, since states Ψ_W are dense in $\mathcal{H}$. Hence we learn that a dense set of operators in $\mathcal{A}'$ are operators that act by $\mathsf{T}\Psi_\mathsf{S} = \mathsf{S}\Psi_\mathsf{W} = \Psi_{\mathsf{S}\mathsf{W}}$ for some $\mathsf{W} \in \mathcal{A}_1$. This means that right multiplication in $\mathcal{A}_1$ by $\mathsf{S} \to \mathsf{S}\mathsf{W}$ gives a dense set of operators in $\mathcal{A}'$. $\mathcal{A}'$ is the closure of this set.

What is happening here is that there are always two commuting algebras that act on an algebra $\mathcal{A}$. $\mathcal{A}$ can act on itself by left multiplication, and $\mathcal{A}$ acting on itself in this way commutes with another algebra $\mathcal{A}'$ that acts on $\mathcal{A}$ by right multiplication. $\mathcal{A}'$ is isomorphic to what is called the opposite algebra of $\mathcal{A}$, sometimes denoted $\mathcal{A}^{\mathrm{op}}$. Elements of $\mathcal{A}^{\mathrm{op}}$ are in one-to-one correspondence with elements of $\mathcal{A}$, but they are multiplied in the opposite order. For $\mathsf{S} \in \mathcal{A}$, write S^{op} for the corresponding element of $\mathcal{A}^{\mathrm{op}}$. Multiplication in $\mathcal{A}^{\mathrm{op}}$ is defined by $\mathsf{S}^{\mathrm{op}}\mathsf{T}^{\mathrm{op}} = (\mathsf{T}\mathsf{S})^{\mathrm{op}}$, which agrees with right multiplication of $\mathcal{A}$ on itself, showing that $\mathcal{A}' \cong \mathcal{A}^{\mathrm{op}}$. The mathematical statement here is called the commutation theorem for semifinite traces. It says that a von Neumann algebra $\mathcal{A}$ with semifinite trace Tr and the opposite algebra $\mathcal{A}^{\mathrm{op}}$ acting on it from the right are commutants on the Hilbert space $\mathcal{H} = \{\mathsf{a} \in \mathcal{A} : \mathrm{Tr}\, \mathsf{a}^\dagger \mathsf{a} < \infty\}$.

If a string S corresponds to an invertible operator (even if the inverse is an unbounded operator affiliated to $\mathcal{A}$ rather than an element of $\mathcal{A}$), the state Ψ_S is cyclic-separating for $\mathcal{A}$ and $\mathcal{A}'$; an example is $\mathsf{S} = e^{-\beta H/2}$ with Ψ_S the thermofield double state.

The intersection $\mathcal{A} \cap \mathcal{A}'$ consists of operators that commute with $\mathcal{A}$ (since they are in $\mathcal{A}'$) and with $\mathcal{A}'$ (since they are in $\mathcal{A}$). So the intersection is the common center of $\mathcal{A}$ and $\mathcal{A}'$. Under hypotheses discussed in section 3.2, this common center is trivial, $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}$. Since $\mathcal{A}$ and $\mathcal{A}'$ are von Neumann algebras that are commutants, a general theorem of von Neumann asserts that the algebra $\mathcal{A} \vee \mathcal{A}'$ generated by $\mathcal{A}$ and $\mathcal{A}'$ together is the whole algebra $B(\mathcal{H})$ of bounded operators on $\mathcal{H}$. We will challenge this claim in section 4 by using baby universes to define what will appear to be operators on $\mathcal{H}$ that commute with both $\mathcal{A}$ and $\mathcal{A}'$. This claim will turn out to fail in an instructive fashion.

To complete the story, we would like to show that the algebras $\mathcal{A}$ and $\mathcal{A}'$ coincide with the algebras $\mathcal{A}_L$ and $\mathcal{A}_R$ that were defined in the Lorentz signature picture in section 3.2. In one direction, this is clear. $\mathcal{A}$ was defined as the smallest von Neumann algebra containing operators that correspond to the strings in eqn. (3.22), acting on the left side of a two-sided system. All these strings correspond to bounded operators built from H and the matter operators Φ. $\mathcal{A}_L$ was defined as the algebra of all bounded operators built from H_L and matter operators Φ_L, acting on the left boundary. So $\mathcal{A} \subset \mathcal{A}_L$. Similarly $\mathcal{A}' \subset \mathcal{A}_R$. Since $\mathcal{A}$ and $\mathcal{A}'$ are commutants (meaning that they are each as large as they can be while commuting with the other), and $[\mathcal{A}_L, \mathcal{A}_R] = 0$, it is impossible for $\mathcal{A}_L$ to be bigger than $\mathcal{A}$ or for $\mathcal{A}_R$ to be bigger than $\mathcal{A}'$. Thus $\mathcal{A}_L = \mathcal{A}$, $\mathcal{A}_R = \mathcal{A}'$.

In this discussion, we started with an algebra $\mathcal{A}_0$ of strings and then we formally defined a state Ψ_S for every $\mathsf{S} \in \mathcal{A}_0$. At this level, then, there is trivially a state for every element $\mathsf{S} \in \mathcal{A}_0$. Then we took a completion of the space generated by the states Ψ_S to get a Hilbert space $\mathcal{H}$, and a completion of $\mathcal{A}_0$ to get the algebra $\mathcal{A}$. One can ask whether after taking completions there is still a Hilbert space state for every element of the algebra. The answer to this question is "no," because the state formally associated to an algebra element x might not be normalizable. For example, as we have already discussed, the state $|\mathbf{1}\rangle$ that would be formally associated to the identity element $\mathbf{1} \in \mathcal{A}$ is not normalizable and so is not an element of $\mathcal{H}$. But this is the only obstruction. Since the norm squared of a state $|\mathsf{x}\rangle$ corresponding to an algebra element x is supposed to satisfy $\langle \mathsf{x}|\mathsf{x}\rangle = \mathrm{Tr}\, \mathsf{x}^\dagger \mathsf{x}$, the necessary condition for the existence of a state $|\mathsf{x}\rangle \in \mathcal{H}$ that corresponds to an algebra element x is simply

$$\mathrm{Tr}\, \mathsf{x}^\dagger \mathsf{x} < \infty. \tag{3.25}$$

If such a state $|\mathsf{x}\rangle$ does exist, then for every $\mathsf{a} \in \mathcal{A}$,

$$\langle \mathsf{x}|\mathsf{a}|\mathsf{x}\rangle = \mathrm{Tr}\, \mathsf{a}\mathsf{x}\mathsf{x}^\dagger. \tag{3.26}$$

This formula says that the density matrix of the state $|\mathsf{x}\rangle$ on $\mathcal{A}_R$ is $\rho = \mathsf{x}\mathsf{x}^\dagger$. If $\mathsf{x} \in \mathcal{A}_0$ is a string, then the string describing $\mathsf{x}\mathsf{x}^\dagger$ is formed by concatenating x with a reversed-ordered copy of itself. Similarly, $\mathrm{Tr}\, \rho^n = \mathrm{Tr}\, (\mathsf{x}\mathsf{x}^\dagger)^n$ is computed by evaluating a Euclidean path integral on a disc with boundary formed by gluing together n copies of $\mathsf{x}\mathsf{x}^\dagger$. It should be clear that the rule we have just described for computing $\mathrm{Tr}\, \rho^n$ using a Euclidean gravitational path integral is exactly the usual rule used in replica trick entropy computations in Euclidean gravity. This rule is usually justified either by appealing to the AdS/CFT dictionary to relate the gravitational path integral to microscopic CFT entropy computations [20, 21] or, in settings where no explicit microscopic theory is known, simply by its success in giving sensible answers [19]. In contrast, we started with an explicit asymptotic boundary algebra $\mathcal{A}_R$ in a canonically quantised gravity theory. We argued that this algebra has (up to an additive constant) a unique definition of entropy. Finally, we showed that, given a state Ψ prepared by some Euclidean path integral, we can compute the entropy of Ψ on the algebra $\mathcal{A}_R$ – in the canonically quantised theory – using the usual rules for replica trick Euclidean gravity computations.

4 Baby Universe "Operators"

Up to this point we have assumed the spacetime topology to be a disc (in Euclidean signature) or a strip (in Lorentz signature). But in a theory of gravity, it is natural to consider more general topologies. An obvious direction, which we explore starting in section 5, is to include wormholes and topology change in the dynamics. First, however, we will consider wormholes and closed baby universes as external probes. Via such probes, we can define what will appear at first sight to be operators with paradoxical properties. The paradox will be resolved in an instructive fashion.

To use wormholes as probes, we adapt to the present context a construction made in [22]. In Euclidean signature, instead of assuming spacetime to have just one asymptotic boundary on which the dual quantum mechanics is defined, we add a second asymptotic boundary that creates a closed baby universe. So the Euclidean spacetime becomes a "double trumpet" (fig. 6(a)). A hyperbolic metric on the double trumpet has a single real

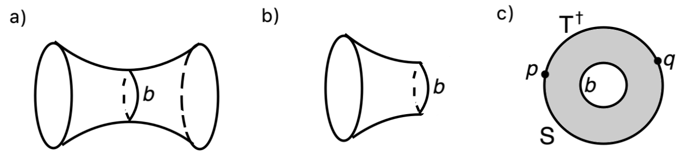

Figure 6. (a) A double trumpet. The boundary quantum mechanics is defined on the left boundary, and the boundary condition on the right boundary is chosen so the internal geodesic has circumference b and the matter fields are in state Λ. (b) Such pictures can be abbreviated by omitting a "trumpet" that connects to the external boundary (in this example, the omitted region is the portion to the right of the closed geodesic). The omitted region can always be glued back in a unique way, and this is always assumed. (c) A face-on view of the "trumpet" in (b), which topologically is an annulus. The inner boundary is a geodesic of circumference b and the quantum mechanics is defined on the outer boundary. The outer boundary has been labeled by strings S, $\mathsf{T}^\dagger$, separated by marked points p, q. The path integral on this Euclidean spacetime computes $\langle\Psi_\mathsf{T}|\mathcal{O}_{b,\Lambda}|\Psi_\mathsf{S}\rangle$.

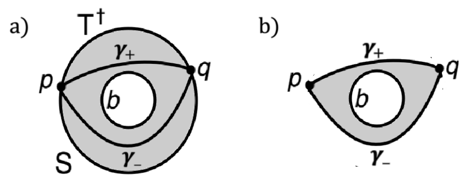

Figure 7. (a) To interpret the path integral in fig. 6(b) as a matrix element $\langle\Psi_\mathsf{T}|O_{b,\Lambda}|\Psi_\mathsf{S}\rangle$, we introduce the indicated geodesics γ_+ and γ_- that go "below" and "above" the hole. (b) The operator $\mathcal{O}_{b,\Lambda}$ is represented by the path integral in the "middle region" Σ_0 between γ_+ and γ_-. The path integral in the region below γ_- prepares the ket $|\Psi_\mathsf{S}\rangle$, and that in the region above γ_+ prepares the bra $\langle\Psi_\mathsf{T}|$.

modulus, namely the circumference b of the simple closed geodesic in its "core." We assume that the boundary quantum mechanics is defined at the "left" end of the double trumpet ($\tau \to -\infty$); thus, at this end a cutoff of the usual type is imposed near the conformal boundary. Along the "right" boundary ($\tau \to +\infty$), which we will call an "external" boundary, we do not place such a cutoff, but instead impose a condition on the asymptotic behavior of the metric which ensures that the circumference of the closed geodesic will be b. Concretely, to do this, we observe that the hyperbolic metric of the double trumpet has a standard form

$$\mathrm{d}s^2 = \mathrm{d}\tau^2 + \cosh^2\tau \mathrm{d}\phi^2, \tag{4.1}$$

with $\phi \cong \phi + b$, $-\infty < \tau < \infty$. The closed geodesic that is homologous to the boundary is at $\tau = 0$ and its circumference is b. Setting $y = \frac{b}{4\pi}e^{-\tau}$, $\sigma = \frac{2\pi}{b}\phi$, the metric takes the form

$$\mathrm{d}s^2 = \frac{\mathrm{d}y^2 + \mathrm{d}\sigma^2 + y^2\frac{b^2}{8\pi^2}\mathrm{d}\sigma^2 + \mathcal{O}(y^4)}{y^2}, \tag{4.2}$$

and we see that b can indeed be encoded in the coefficient of a subleading term of the metric near the conformal boundary at $y = 0$. In pure JT gravity, the boundary condition that we want on the external boundary can be defined just by fixing a value of b. In JT gravity coupled to matter, we additionally need a boundary condition on the matter fields. Such a boundary condition can be determined by any rotation-invariant state Λ in the closed universe matter Hilbert space $\mathcal{H}_{\rm cl}^{\rm matt}$. By rotation invariance, we mean invariance under $\phi \to \phi + \text{constant}$.[14]

We can slightly simplify the following by "cutting" along the closed geodesic at $\tau = 0$ and discarding the "exterior" piece ($\tau > 0$). Upon doing so, Σ becomes an ordinary trumpet (rather than a double trumpet) with a quantum boundary at big distances and a closed geodesic boundary of circumference b on which the matter state Λ is inserted (fig. 6(b)). Such truncations also exist and are convenient in cases discussed later with multiple external boundaries. In what follows, we always draw the truncated version of the spacetime.

In section 3.3, given a pair of strings S, T, we defined inner products $\langle \Psi_{\mathsf{T}}, \Psi_{\mathsf{S}} \rangle$ via a path integral on a disc with its boundary labeled by $\mathsf{T}^\dagger \mathsf{S}$. An obvious idea now is to consider a similar path integral on a Riemann surface Σ that is a disc with a hole labeled by some b, Λ (fig. 6(c)). Σ has two marked points on its outer boundary, namely the endpoints p, q of the seqment labeled by S.

A natural expectation is that this path integral can be interpreted as $\langle \Psi_{\mathsf{T}} | \mathcal{O}_{b,\Lambda} | \Psi_{\mathsf{S}} \rangle$ for some operator $\mathcal{O}_{b,\Lambda}$. To justify this expectation, we note (fig. 7(a)) that the points p, q are connected by a unique embedded geodesic γ_- that goes "below" the hole and also by a unique embedded geodesic γ_+ that goes "above" the hole. These geodesics exist because on a hyperbolic two-manifold, there is always a unique geodesic in each homotopy class of paths; this is a fact that we will use repeatedly. Correspondingly, Σ is the union of a portion Σ_- below γ_-, a portion Σ_+ above γ_+, and a portion Σ_0 in between. The path integral on Σ_- computes $|\Psi_{\mathsf{S}}\rangle$ and the path integral on Σ_+ computes $\langle \Psi_{\mathsf{T}}|$, so to interpret the path integral on Σ as a matrix element $\langle \Psi_{\mathsf{T}} | \mathcal{O}_{b,\Lambda} | \Psi_{\mathsf{S}} \rangle$, the operator $\mathcal{O}_{b,\Lambda}$ has to be represented by the path integral on Σ_0. In fact, let ℓ_- and ℓ_+ be the renormalized lengths of the geodesics γ_- and γ_+. The path integral on Σ_0 with specified values of ℓ_- and ℓ_+ computes the kernel $\mathcal{O}_{b,\Lambda}(\ell_+, \ell_-)$ in the length basis. This kernel is an operator acting on the matter Hilbert space $\mathcal{H}_{\rm cl}^{\rm matt}$, though we do not indicate that explicitly in the notation.

The renormalized lengths ℓ_+ and ℓ_- are not uniquely determined by the complex structure of Σ_0; they depend also on the positions of the boundary particles or in other words on the cutoffs near the boundary points p, q (the cutoff variables were called σ_L, σ_R or χ_L, χ_R in section 2.1). However, the difference $\Delta\ell = \ell_+ - \ell_-$ does not depend on the cutoffs and is a modulus of the Riemann surface Σ_0. In fact, this is the only modulus of Σ_0. In general, a hyperbolic disc with n marked points on the boundary and m holes with geodesic boundaries (each of specified circumference) depends on $n + 2m - 3$ moduli. In the case of Σ_0, one has $n = 2$, $m = 1$, so there is only 1 modulus and it is $\Delta\ell$. This means that when we compute the kernel $\mathcal{O}_{b,\Lambda}(\ell_+, \ell_-)$, for specified values of ℓ_+, ℓ_-, the moduli of

[14] If Λ is not invariant under shifts of ϕ, then in doing the path integral on the double trumpet, we will have to integrate over a twist of the left of the double trumpet relative to the right, and this will effectively replace Λ by its rotation-invariant projection. In section 5.2, we describe the closed universe Hilbert space in a more leisurely fashion.

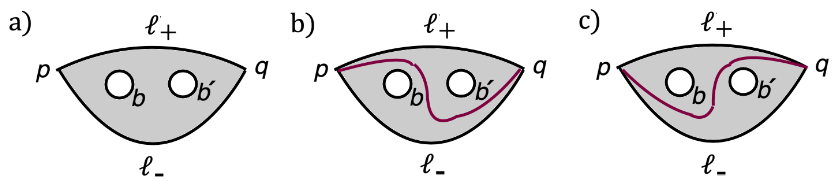

Figure 8. (a) The configuration appropriate to computing the kernel, in the length basis, of a product of baby universe operators. The upper and lower boundaries are geodesics that have been labeled by their renormalized lengths ℓ_+ and ℓ_-. (b) and (c) In a hyperbolic manifold, there is a unique geodesic in any prescribed homotopy class. So in particular there exists a unique geodesic from p to q that goes above one chosen hole and below the other, as shown here in purple.

the surface and the positions of the boundary particles can all be considered fixed.[15] So we compute $\mathcal{O}_{b,\Lambda}(\ell_+, \ell_-)$ by integrating over matter fields on a fixed Riemann surface. This calculation is completely well-defined, with no infrared or ultraviolet divergence. So there is no difficulty in defining matrix elements of $\mathcal{O}_{b,\Lambda}$ between states defined by strings.

Having understood how to represent a "baby universe" operator $\mathcal{O}_{b,\Lambda}$ by a path integral, a next step is to try to represent a product of such operators $\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda}$. An obvious guess is to add one more hole to the Riemann surface Σ_0, giving the Riemann surface Σ_1 sketched in fig. 8(a). To see that this guess is correct, we note (figs. 8(b,c)) that in Σ_1, the boundary points p, q are connected by a unique geodesic γ_0 that goes above any one chosen hole and below the other. This means, depending on whether we consider fig. 8(b) or 8(c), that the path integral on Σ_1, for fixed values of ℓ_+, ℓ_-, computes the kernel of the product $\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda}$ or of the product $\mathcal{O}_{b,\Lambda}\mathcal{O}_{b',\Lambda'}$. Since the same path integral computes either of these products, we appear to learn that these operators commute:

$$\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda} = \mathcal{O}_{b,\Lambda}\mathcal{O}_{b',\Lambda'}. \tag{4.3}$$

It is instructive to count the moduli of the surface Σ_1. We can do this based on either fig. 8(b) or 8(c). There are now three renormalized lengths ℓ_+, ℓ_0, and ℓ_-, of the upper, middle, and lower geodesic in the figure. The two differences $\ell_+ - \ell_0$ and $\ell_0 - \ell_-$ do not depend on the positions of the boundary particles and are moduli. There is one more real modulus, which corresponds to the fact that we can cut Σ_1 along γ_0, slide the upper part by an arbitrary amount relative to the lower part, and then reglue. Starting with a hyperbolic metric on Σ_1, this operation gives another hyperbolic metric, so the amount of sliding is a third modulus. The moduli that we have described are the only ones, since the formula $n + 2m - 3$, with $n = m = 2$, predicts 3 moduli.

When one computes the kernel $(\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda})(\ell_+, \ell_-)$ for given ℓ_+, ℓ_-, one modulus $\ell_+ - \ell_-$ is given, but one has to integrate over the other two moduli. This integration is harmless in JT gravity without matter, but it turns out that it diverges in JT gravity with matter. Why this happens and why it is important will be explained presently. For now, however,

[15] More precisely, we can fix the position of the left boundary particle by a gauge condition and determine the position of the right boundary particle and the modulus of Σ_0 to match the desired values of ℓ_+, ℓ_-.

we proceed formally. Let λ be the sliding modulus and let T_λ be the operator that shifts a state by an amount λ (thus $T_\lambda = e^{\mathrm{i}\lambda P}$ where P is the momentum operator, a generator of $SL(2,\mathbb{R})$). Since the two undetermined moduli are the length parameter ℓ_0 of γ_0 and the sliding modulus, integration over those moduli leads to a formula

$$(\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda})(\ell_+,\ell_-) = \int_{-\infty}^{\infty} \mathrm{d}\ell_0 \mathrm{d}\lambda \, \mathcal{O}_{b',\Lambda'}(\ell_+,\ell_0) T_\lambda \mathcal{O}_{b,\Lambda}(\ell_0,\ell_-). \tag{4.4}$$

The integral over ℓ_0 is part of the usual sum over intermediate states in the matrix element of a product of operators. But what should we make of the integral over λ?

Here we must remember that the Hilbert space of JT gravity, with or without matter, is most naturally defined as a space of $\widetilde{SL}(2,\mathbb{R})$ coinvariants, not invariants. When one takes the inner product of two states defined as coinvariants, one must include an integral over the group action, as in eqn. (2.28). For the same reason, in constructing the kernel that represents a product of operators from the kernels that represent the individual operators, one needs to include an integral over the group action. However, in the present context, since the boundary points p, q are fixed, the operators $\mathcal{O}_{b,\Lambda}$, $\mathcal{O}_{b',\Lambda'}$ act trivially on the variables that were called T_L, T_R in section 2.1. Given this, two of the $\widetilde{SL}(2,\mathbb{R})$ integrals become trivial and the integral over $\widetilde{SL}(2,\mathbb{R})$ reduces to the integral over the sliding modulus λ that appears in eqn. (4.4).

It may seem reasonable to conclude that the "baby universe" operators $\mathcal{O}_{b,\Lambda}$ commute with each other. One might then think that their joint eigenvalues would be the classical α-parameters [24, 25].

However, let us go on and consider also the boundary algebras. For example, let S be a string, viewed as an element of the left boundary algebra $\mathcal{A}_L$. How would we compute a product of operators $\mathsf{S}\mathcal{O}_{b,\Lambda}$ or $\mathcal{O}_{b,\Lambda}\mathsf{S}$? The obvious way is to add a hole to the Riemann surface that we would use to compute a matrix element of S (fig. 5(a)). The candidate spacetime to compute $\mathsf{S}\mathcal{O}_{b,\Lambda}$ or $\mathcal{O}_{b,\Lambda}\mathsf{S}$ is shown in fig. 9(a). We note immediately that there is no natural ordering between S and $\mathcal{O}_{b,\Lambda}$ in this spacetime, so if $\mathsf{S}\mathcal{O}_{b,\Lambda}$ and $\mathcal{O}_{b,\Lambda}\mathsf{S}$ can be computed in this fashion, the operators in question must commute. That is in fact the case, as we see by drawing an appropriate geodesic above or below the hole (figs. 9(b,c)). With one choice of geodesic, one learns that the path integral on this surface computes $\mathcal{O}_{b,\Lambda}\mathsf{S}$; with the other choice, one learns that the same path integral computes $\mathsf{S}\mathcal{O}_{b,\Lambda}$. The reasoning here is similar to what we have seen in discussing the product of two baby universe operators and further details are left to the reader. We just note that in this case, there is no problem with the integral over moduli, whether matter is present or not.

Likewise, the baby universe operators $\mathcal{O}_{b,\Lambda}$ commute with $\mathcal{A}_R$.

In JT gravity without matter, all this is true. In the absence of matter fields, the baby universe operators are labeled just by the length parameter b. The kernel that describes $\mathcal{O}_b$ in the length basis can indeed be obtained from the path integral of fig. 7(b). This path integral was evaluated in [34], with the result that in the absence of matter, $\mathcal{O}_b$ is a function of the boundary Hamiltonian (either H_L or H_R, since they are equal). Any possible boundary operator is also a function of the boundary Hamiltonian, since this

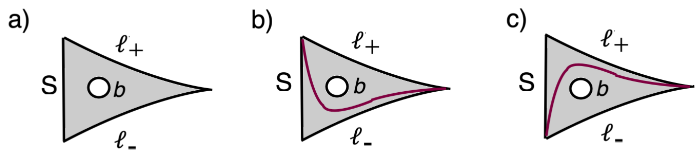

Figure 9. (a) Adding a geodesic hole of circumference b to the configuration of fig. 5(a), we get a Riemann surface that is a candidate for describing the kernel of the operator product $\mathsf{S}\mathcal{O}_{b,\Lambda}$ or $\mathcal{O}_{b,\Lambda}\mathsf{S}$ in the length basis. (b) and (c) By considering a geodesic that starts at the upper or lower corner of the diagram and goes below or above the hole, we decompose the picture of (a) in two pieces that respectively describe S and $\mathcal{O}_{b,\Lambda}$. This confirms that the picture in (a) does compute this operator product and that $\mathsf{S}\mathcal{O}_{b,\Lambda} = \mathcal{O}_{b,\Lambda}\mathsf{S}$.

operator generates the algebra of boundary observables. Hence the baby universe operators commute with each other and with $\mathcal{A}_L$ and $\mathcal{A}_R$.

In the presence of matter, however, we have a problem. We have argued in section 3 that in the presence of matter, $\mathcal{A}_L$ and $\mathcal{A}_R$ have trivial centers and are each other's commutants. This implies, in particular, that there are no operators (other than c-numbers) that commute with both $\mathcal{A}_L$ and $\mathcal{A}_R$. So if our previous claims are correct, something must be wrong with the assertion that there are baby universe operators $\mathcal{O}_{b,\Lambda}$ that commute with $\mathcal{A}_L$ and $\mathcal{A}_R$.

It appears that what is wrong is the assertion that the objects $\mathcal{O}_{b,\Lambda}$ make sense as Hilbert space operators. They do make sense as quadratic forms, meaning that they have well-defined matrix elements between a dense set of Hilbert space states. For example, for strings S, T, there is no problem in defining the matrix element $\langle \Psi_{\mathsf{T}} | \mathcal{O}_{b,\Lambda} | \Psi_S \rangle$. However, to define a Hilbert space operator, one needs more. If an object $\mathcal{O}$ is supposed to act as an operator on a Hilbert space $\mathcal{H}$, there should be at a minimum a dense set of states $\Psi \in \mathcal{H}$ such that $\mathcal{O}\Psi$ can be defined as a vector in $\mathcal{H}$. This condition requires $|\mathcal{O}\Psi|^2 < \infty$ or $\langle \Psi | \mathcal{O}^\dagger \mathcal{O} | \Psi \rangle < \infty$. So in order for $\mathcal{O}_{b,\Lambda}$ to be defined as a Hilbert space operator, the product $\mathcal{O}^\dagger_{b,\Lambda}\mathcal{O}_{b,\Lambda}$ should have a finite expectation value in a dense set of states. In fact, the adjoint of $\mathcal{O}_{b,\Lambda}$ is $\mathcal{O}_{b,\bar{\Lambda}}$, where $\bar{\Lambda}$ is the CPT conjugate of Λ. So we need the product $\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}$ to have finite matrix elements, in a dense set of states.

But matrix elements of the product $\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}$ diverge because of the moduli space integration that we encountered in eqn. (4.4). To see this, we simply look at the moduli space integral of fig. 8 in what one might call the "closed universe channel." In the Riemann surface Σ_1 with two holes that we used to study products $\mathcal{O}_{b',\Lambda'}\mathcal{O}_{b,\Lambda}$, there is a unique embedded geodesic γ^* that goes around both holes (fig. 10(a)). One can interpret the two moduli of Σ_1 that remain when $\ell_+ - \ell_-$ is specified as the circumference b^* of γ^*, and the angle of a possible "twist"[16] around γ^*. In particular, b^* has to be integrated from 0 to ∞. But in any conventional two-dimensional QFT, the path integral on this surface will diverge at $b^* \to 0$ because the ground state energy of a two-dimensional field theory

[16]The twist – known as a Dehn twist – is made by cutting along γ^*, rotating one side by some angle relative to the other, and then regluing.

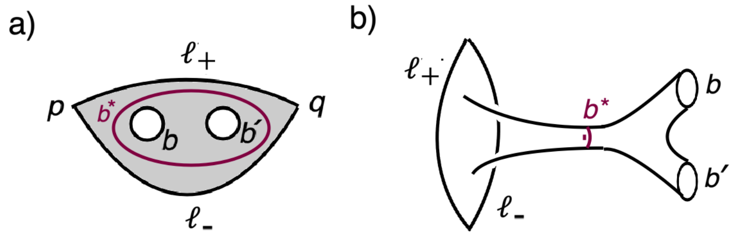

Figure 10. (a) A different way to decompose the Riemann surface (fig. 8(a)) that describes the product of two baby universe operators in the length basis. Shows in purple is the unique embedded geodesic γ^*, of circumference b^*, that encloses both holes. (b) A different way to look at the configuration in (a). The purpose of this picture is to make it obvious that the path integral will diverge for $b^* \to 0$, because of a negative energy state propagating down the long tube that separates the left and right of the figure.

on a small circle is always negative. For example, a CFT with central charge c on a circle of circumference b^* has a ground state energy $-\pi c/6b^*$. Any QFT that is conformally invariant at short distances similarly has a negative ground state energy on a small circle. For small b^*, the hyperbolic metric on Σ_1 has a very long tube separating the part of Σ_1 where the dual quantum mechanics is defined from the two holes (fig. 10(b)). Propagation of a negative energy ground state down that long tube will give a contribution that grows exponentially for $b^* \to 0$, leading to a divergence in the path integral that describes the product of two baby universe "operators." In the language of fig. 8 and eqn. (4.4), this is a divergence when $\ell_0 \to +\infty$, because a geodesic that goes above one of the two holes and below the other in fig. 8 must in fig. 10(b) go all the way up the tube and back again, so its length diverges for $b^* \to 0$.

The conclusion, then, is that the objects $\mathcal{O}_{b,\Lambda}$ make sense as quadratic forms, with well-defined matrix elements between suitable states, but they do not make sense as Hilbert space operators. In particular, the $\mathcal{O}_{b,\Lambda}$ do not have eigenvectors and eigenvalues. If Ψ were an eigenvector of $\mathcal{O}_{b,\Lambda}$, say with eigenvalue w, then we would have $\langle\Psi|\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}|\Psi\rangle = |\mathcal{O}_{b,\Lambda}\Psi|^2 = |w|^2|\Psi|^2 < \infty$, contradicting the universal nature of the divergence in the $\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}$ product.[17] Thus, in JT gravity coupled to matter, one cannot define α parameters as eigenvalues of the $\mathcal{O}_{b,\Lambda}$.

Since this phenomenon may seem unfamiliar, we will mention an elementary situation in which something similar occurs. Let ϕ be a local operator in some quantum field theory in Minkowski spacetime M of any dimension $D \geq 2$. Consider two complementary Rindler wedges in M, with respective operator algebras $\mathcal{A}_L$, $\mathcal{A}_R$. According to the Bisognano-Wichman theorem [35], $\mathcal{A}_L$ and $\mathcal{A}_R$ have trivial center and are commutants. We can reach an apparent contradiction as follows. Let p be a point in the bifurcation surface where the two wedges meet and consider the "operator" $\phi(p)$. One can formally argue that $\phi(p)$

[17]It does not help to assume that $\mathcal{O}_{b,\Lambda}$ has a continuous spectrum. Let Π be the projection operator onto states with $|\mathcal{O}_{b,\Lambda}| \leq w$, and let Ψ be in the image of Π. Then we would have $|\langle\Psi|\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}|\Psi\rangle| \leq |w|^2|\Psi|^2$, again contradicting the universal nature of the divergence in $\mathcal{O}_{b,\bar{\Lambda}}\mathcal{O}_{b,\Lambda}$.

commutes with[18] $\mathcal{A}_L$ and $\mathcal{A}_R$, seemingly contradicting the theorem. The resolution is that $\phi(p)$ makes sense as a quadratic form, since it has well-defined matrix elements between a suitable dense set of states, but does not make sense as an operator. For example, in free field theory, $\phi(p)$ has well-defined matrix elements between Fock space states. However, $\phi(p)$ does not make sense as an operator and does not have eigenvalues and eigenvectors, because if Ψ is any Hilbert space state, $\phi(p)\Psi$ is unnormalizable. The norm squared of $\phi(p)\Psi$ would equal $\lim_{q\to p}\langle\Psi|\phi^\dagger(q)\phi(p)|\Psi\rangle$, and this is divergent because the product $\phi^\dagger(q)\phi(p)$ is singular for $q \to p$.

The $\mathcal{O}_{b,\Lambda}$ would presumably be far more significant if they could be defined as operators, for then their eigenvalues ("α-parameters") could be used to decompose the Hilbert space. Note that "operators" $\phi(p)$ on the bifurcation surface are not very useful in studying physics in the Rindler wedge.

Even though the $\mathcal{O}_{b,\Lambda}$ do not make sense as operators, because they make sense as quadratic forms one might still worry about an apparent conflict with our claim in section 3 that the trace on a von Neumann factor is unique up to rescaling. Specifically, for each b, Λ, we can define a new "trace" $\mathrm{Tr}_{b,\Lambda}(\mathsf{a})$ by evaluating a Euclidean path integral on the annulus shown in fig. 6 with boundary conditions at infinity defined using a as in section 3.3. By construction, this satisfies $\mathrm{Tr}_{b,\Lambda}(\mathsf{a}\mathsf{a}') = \mathrm{Tr}_{b,\Lambda}(\mathsf{a}'\mathsf{a})$ for all $\mathsf{a}, \mathsf{a}' \in \mathcal{A}$. However, in contrast to the usual trace, there is no formal argument based on reflection positivity of the bulk path integral that $\mathrm{Tr}_{b,\Lambda}$ is positive on positive operators, and that is actually not true. If Λ is the CFT ground state, then from eqn. (3.30) of [34], one can deduce that for any function $f(H)$ of the Hamiltonian H, $\mathrm{Tr}_{b,\Lambda}\, f(H) = \int_0^\infty \mathrm{d}E\, f(E)\frac{\cos(b\sqrt{2E})}{\pi\sqrt{2E}}$ (up to a constant factor that depends on the regularization of the matter path integral), showing the lack of positivity. The same formula actually applies for any Λ, since there is actually no coupling between the gravitational sector and the matter sector in the path integral that computes $\mathrm{Tr}_{b,\Lambda}\, f(H)$.

For any operator $\mathsf{a} \in \mathcal{A}$, the operator $e^{-\varepsilon H}\mathsf{a}$ has a finite trace $\mathrm{Tr}_{b,\Lambda}(e^{-\varepsilon H}\mathsf{a})$. Since $e^{-\varepsilon H}\mathsf{a}$ converges to a as $\epsilon \to 0$, the trace $\mathrm{Tr}_{b,\Lambda}$ (just like the disc trace Tr defined in section 3) is finite on a dense set of operators. And it is not related to Tr by a rescaling.

To understand what is going on here, we need to be a bit more precise. A Type I or II von Neumann factor has a trace that is unique if one requires it to be normal and semifinite; more technically, one says that such a factor has a unique semifinite normal tracial weight. A weight is a linear map ϕ from the positive elements[19] of the algebra $\mathcal{A}$ to $[0, \infty]$. It is tracial if $\phi(\mathsf{ab}) = \phi(\mathsf{ba})$ for all $\mathsf{a}, \mathsf{b} \in \mathcal{A}$. We will briefly discuss semifiniteness at the end of this section. The important qualification for our purposes is that uniqueness depends

[18] $\mathcal{A}_L$ or $\mathcal{A}_R$ is generated by functions of smeared quantum fields. The smearing functions are smooth functions supported in the Rindler wedge. The interior of the Rindler wedge is spacelike separated from the point p, so a nonzero commutator of such a smeared field with $\phi(p)$ must arise from a contribution on the boundary of the Rindler wedge. There is no such contribution, since a smooth function with support in the Rindler wedge vanishes to all orders near the boundary, killing any singularity that commutators of quantum fields may have along the diagonal or at null separation.

[19] A weight is defined only for positive elements to avoid difficulties that one would encounter with $\infty - \infty$ if one attempts in an infinite von Neumann algebra to extend the definition of a typical weight to indefinite elements.

on the trace being normal. Being "normal" is roughly a condition of continuity, but in an infinite von Neumann algebra, this has to be stated with care. A precise definition is that a weight ϕ is normal if given an increasing sequence of positive operators $\mathsf{a}_n \in \mathcal{A}$ that converge to a, we have $\lim_{n\to\infty} \phi(\mathsf{a}_n) = \phi(\mathsf{a})$. The reason for requiring the sequence a_n to be increasing is that in an infinite von Neumann algebra – Type I_∞ or Type II_∞ – one can have, for example, projection operators p_n of arbitrarily large trace. So for a normal weight ϕ, one could have a sequence of positive operators, say $\mathsf{a}_n = \mathsf{a} + p_n/n$, with $\lim_{n\to\infty} \mathsf{a}_n = \mathsf{a}$ but $\lim_{n\to\infty} \phi(\mathsf{a}_n) > \phi(\mathsf{a})$. This is described by saying that ϕ is lower semicontinuous; it can jump downward but not upward in a limit. In the case of an increasing sequence, lower semicontinuity becomes ordinary continuity.

An obvious example of a normal weight is the functional $\phi(\mathsf{a}) = \langle\Psi|\mathsf{a}|\Psi\rangle$, where Ψ is any vector in a Hilbert space $\mathcal{H}$ on which the algebra $\mathcal{A}$ acts. A positive functional of this kind (for any choices of Ψ and $\mathcal{H}$) is said to be "ultraweakly continuous." A function $f : \mathbb{R} \to \mathbb{R}$ is lower semicontinuous if and only if it can be written the limit of a monotonically increasing sequence of continuous functions f_n. Similarly, a weight ϕ is normal if and only if it can be written as the limit of a sequence of monotonically increasing ultraweakly continuous weights $\phi_n(\mathsf{a}) = \langle\Psi_n|\mathsf{a}|\Psi_n\rangle$.

A Type II_∞ factor does have additional densely defined traces – such as we are finding with $\mathrm{Tr}_{b,\Lambda}$ – if one drops the conditions of normality and semifiniteness. For example the tensor product of a Dixmier trace on a Type I_∞ factor with the standard trace on a Type II_1 factor gives a trace on a Type II_∞ factor that is densely defined and positive, but not normal. This example does not seem very similar to our $\mathrm{Tr}_{b,\Lambda}$, however.

The uniqueness statement about traces is simpler in the case of an algebra of Type II_1, because then there is an upper bound on the trace of a projector. In our context, we can transfer the discussion to an algebra of Type II_1 by introducing the projection operator P_0 onto states with energy less than some large cutoff energy E_0. Such a projection arose naturally in [31] in the analysis of de Sitter space. As in that case, the projected algebra $\widetilde{\mathcal{A}} = P_0\mathcal{A}P_0$ is of Type II_1. To prove this, one just observes that P_0 is the identity in $\widetilde{\mathcal{A}}$, and $\mathrm{Tr}\, P_0 < \infty$, showing that $\widetilde{\mathcal{A}}$ is of Type II_1, not II_∞.

A theorem about Type II_1 factors[20] asserts that the usual trace is the unique tracial, ultraweakly continuous linear functional on the algebra. Note that there is no assumption here that the functional must be positive. There is also no analog of semifiniteness; instead the trace is assumed to be defined for all elements of the algebra. A general (not necessarily positive) linear functional ϕ on an algebra $\mathcal{A}$ is ultraweakly continuous if we can write it as $\phi(\mathsf{a}) = \langle\chi|\mathsf{a}|\Psi\rangle$, where χ, Ψ are vectors in some Hilbert space on which the algebra $\mathcal{A}$ acts. In our problem, for the usual trace, we can take $\mathcal{H}$ to be the usual Hilbert space and assume $\chi = \Psi$. The state

$$\Psi_0 = \lim_{\beta\to 0} P_0 \Psi_{\mathrm{TFD}}(\beta) \tag{4.5}$$

[20]For example, see Corollary 6.1.19 in [36].

is normalizable. If a_0 is an element of $\widetilde{\mathcal{A}}$, then, as $\mathsf{a}_0 = P_0 \mathsf{a}_0 P_0$,

$$\begin{aligned} \operatorname{Tr} \mathsf{a}_0 &= \lim_{\beta \to 0} \langle \Psi_{\mathrm{TFD}}(\beta) | \mathsf{a}_0 | \Psi_{\mathrm{TFD}}(\beta) \rangle \\ &= \lim_{\beta \to 0} \langle \Psi_{\mathrm{TFD}}(\beta) | P_0 \mathsf{a}_0 P_0 | \Psi_{\mathrm{TFD}}(\beta) \rangle = \langle \Psi_0 | \mathsf{a}_0 | \Psi_0 \rangle. \end{aligned} \tag{4.6}$$

So the usual trace Tr is indeed ultraweakly continuous on $\widetilde{\mathcal{A}}$.

What about $\operatorname{Tr}_{b,\Lambda}$? One can try to write $\operatorname{Tr}_{b,\Lambda}(\mathsf{a}_0) = \langle \Psi_{b,\Lambda} | \mathsf{a}_0 | \Psi_0 \rangle$ where

$$\Psi_{b,\Lambda} = \lim_{\beta \to 0} P_0 \Psi_{b,\Lambda}(\beta) \tag{4.7}$$

and $\Psi_{b,\Lambda}(\beta)$ is prepared using a path integral on an annulus with an asymptotic boundary of renormalized length $\beta/2$, a geodesic boundary on which the state is defined, and an additional closed geodesic boundary labeled by (b, Λ). However, $\langle \Psi_{b,\Lambda} | \Psi_{b,\Lambda} \rangle$ diverges for exactly the same reason that $\mathcal{O}_{b,\Lambda}$ does not make sense as an operator. So $\Psi_{b,\Lambda}$ is not a Hilbert space state and we do not succeed in proving that $\operatorname{Tr}_{b,\Lambda}$ is ultraweakly continuous. Hence there is no contradiction with $\widetilde{\mathcal{A}}$ being a Type II_1 factor.

We conclude by demonstrating that the trace Tr that we defined originally does indeed satisfy all the expected properties of the standard trace on the full Type II_∞ factor $\mathcal{A}$. We already know that it is tracial. To see that it is normal, we note that the functional $F_\beta(\mathsf{a}) = \langle \Psi_{\mathrm{TFD}}(\beta) | \mathsf{a} | \Psi_{\mathrm{TFD}}(\beta) \rangle$ increases as β tends to zero (since $\partial_\beta e^{-\beta E} \leq 0$ for $E \geq 0$), so normality of the trace follows from $\operatorname{Tr} \mathsf{a} = \lim_{\beta \to 0} \langle \Psi_{\mathrm{TFD}}(\beta) | \mathsf{a} | \Psi_{\mathrm{TFD}}(\beta) \rangle$. Finally, a weight is semifinite if for every nonzero positive operator $\mathsf{a} \in \mathcal{A}$ there exists a positive operator $\mathsf{a}' \leq \mathsf{a}$ such that $\operatorname{Tr}[\mathsf{a}']$ is finite. For any positive $\mathsf{a} \in \mathcal{A}$, the operator $\mathsf{a}^{1/2} P_0 \mathsf{a}^{1/2} \leq \mathsf{a}$ converges to a in the strong operator topology as $E_0 \to \infty$; consequently $\mathsf{a}^{1/2} P_0 \mathsf{a}^{1/2}$ is nonzero for sufficiently large E_0. Since

$$\operatorname{Tr}[\mathsf{a}^{1/2} P_0 \mathsf{a}^{1/2}] = \operatorname{Tr}[P_0 \mathsf{a} P_0] = \langle \Psi_0 | \mathsf{a} | \Psi_0 \rangle \tag{4.8}$$

is finite, this gives us the desired result.

5 Wormhole Corrections

5.1 Overview

We will now explore what happens when one includes wormhole corrections to the analysis of section 3.3. In other words, we will allow for the possibility that spacetime is a (connected) oriented two-manifold M with a specified boundary but otherwise with any topology.[21] This means that the open universe Hilbert space that we have considered so far will be extended by including a Fock space of closed baby universes, as considered by a number of previous authors [22, 25].

[21] We do not assume time-reversal symmetry or equivalently (by the two-dimensional version of the CPT theorem) spatial reflection symmetry. If one does assume such symmetry, one should allow the possibility that M is unorientable. On an unorientable two-manifold, the contribution of very small cross-caps is such that the path integral of JT gravity is divergent, even in the absence of matter [37]. This divergence is somewhat analogous to the small b divergence that occurs in the presence of matter in the orientable case.

In JT gravity coupled to matter, we do not have a framework to discuss the wormhole contributions nonperturbatively. In JT gravity without matter, the matrix model gives such a framework, which has been exploited very successfully for some purposes [38, 39], though it is not clear whether it is useful for the sort of questions that we consider in the present paper.

Still, whether matter is present or not, we can certainly study wormholes order by order in an expansion in the genus, which is a non-negative integer g. In JT gravity with or without matter, a genus g contribution is suppressed relative to the $g = 0$ contribution by a factor e^{-2gS}, where S is the the black hole entropy. Assuming the classical contribution S_0 to the black hole entropy is large, S is large except at extremely low temperatures or energies, where the theory becomes strongly coupled and the genus expansion will break down. At moderate or high temperatures, the genus expansion is a reasonable framework for studying wormhole contributions, and we will work in that framework.

The wormhole expansion in JT gravity coupled to matter has a well-known technical problem, which we already encountered in section 4. In a path integral with a dynamical wormhole, one will have to integrate over the circumference b of the wormhole, and the integral will diverge for $b \to 0$, because the ground state energy of a quantum field theory on a circle is negative and of order $-1/b$ (it is $-\pi c/6b$ for a CFT with central charge c). We will proceed formally, ignoring this issue. Since the divergence at small b is an ultraviolet issue, one can reasonably hope that JT gravity with matter is an approximation to a better theory in which our general considerations are applicable and the wormhole contributions are convergent. Of course, in section 4, the divergence at $b \to 0$ was crucial to the left and right boundary operator algebras being commutants with trivial center. How that story is modified (or not) in a regulated theory with wormholes potentially depends on the details of the regulated theory. We discuss some possibilities at the end of section 5.6.

It is also true that the divergence in the wormhole amplitudes does not arise for JT gravity without matter. But we do not want to be limited to that case, since in that case the algebra of boundary observables is commutative and not so interesting.

In section 5.2, we discuss from a bulk point of view the Hilbert space of JT gravity coupled to matter in the presence of closed universes. In section 5.3, we analyze the natural Hilbert space from a boundary point of view. Purely from a boundary point of view, it is straightforward to include wormhole contributions to path integrals and thereby to generalize the definitions of a trace, a Hilbert space, and an algebra of observables that were given in section 3.3. In sections 5.4-5.5, we make contact between the boundary analysis and the bulk analysis. This is not nearly as straightforward as it was in section 3.3 in the absence of wormholes.

What we learn can be summarized as follows. With wormholes included, the algebra of boundary observables is modified but is still of Type II_∞. In the theory with wormholes, the natural boundary Hilbert space $\mathcal{H}_{\mathrm{bdry}}$ is a small and hard to characterize subspace of a much bigger bulk Hilbert space $\mathcal{H}_{\mathrm{bulk}}$. However, the difference is undetectable by a boundary observer, in the sense that every pure or mixed state on $\mathcal{H}_{\mathrm{bulk}}$ is equivalent, for a boundary observer, to some state on $\mathcal{H}_{\mathrm{bdry}}$. In fact, the state on $\mathcal{H}_{\mathrm{bdry}}$ can be assumed to be pure. Roughly, not being able to see beyond the horizon, a boundary observer cannot detect the extra degrees of freedom described by $\mathcal{H}_{\text{bulk}}$.

5.2 The Hilbert Space From A Bulk Point Of View

In Lorentz signature, a (connected) closed universe with constant scalar curvature $R = -2$ can be described by the metric

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \cos^2\tau\,\mathrm{d}\phi^2, \tag{5.1}$$

where $\phi \cong \phi + b$, with an arbitrary $b > 0$. Thus ϕ parametrizes a circle S_ϕ. We will call this spacetime U_b. U_b has a big bang singularity at $\tau = -\pi/2$, and a big crunch at $\tau = \pi/2$. The interpretation of those singularities in quantum theory is obscure, to say the least, but they will not be too troublesome for the issues addressed in the present article.

We would like to describe a Hilbert space of quantum states for JT gravity possibly coupled to matter in such a closed universe. This is straightforward. The only modulus of the closed universe is b. Quantizing the matter system on U_b gives a Hilbert space $\mathcal{H}^{\text{matt}}_{\text{cl},b}$. The isometry group of the closed universe is just the group $U(1)_\phi$ of constant shifts of ϕ. We have to impose $U(1)_\phi$ as a group of constraints. Let P be the generator of $U(1)_\phi$ (the operator that measures the momentum around S_ϕ). Since the group $U(1)_\phi$ is compact, imposing the constraint means simply restricting to the subspace of $\mathcal{H}^{\text{matt}}_{\text{cl},b}$ with $P = 0$. We will call this subspace $\mathcal{H}^{\text{matt}}_{\text{cl}_0,b}$.

In addition, we have to take into account the gravitational sector. The only dynamical variables of JT gravity in this closed universe are b and its canonical momentum. Therefore, in addition to its dependence on the matter variables, a quantum state is a function of b.

Thus finally we can describe the Hilbert space $\mathcal{H}_{\text{cl}}$ produced by quantizing JT gravity coupled to matter in a closed universe. A general state $\Psi \in \mathcal{H}_{\text{cl}}$ can be represented by a function $\psi(b)$ that is valued in $\mathcal{H}^{\text{matt}}_{\text{cl}_0,b}$. Inner products of such states are defined by integration over b along with the natural inner product in the matter sector. So if Ψ_1, Ψ_2 correspond to functions $\psi_1(b), \psi_2(b)$, then

$$\langle \Psi_1, \Psi_2 \rangle = \int_0^\infty \mathrm{d}b\,\langle \psi_1(b), \psi_2(b) \rangle. \tag{5.2}$$

If the matter theory is conformally invariant, then $\mathcal{H}_{\text{cl}}$ can be described more simply. In that case, $\mathcal{H}^{\text{matt}}_{\text{cl},b}$ is independent of b, and we denote its $P = 0$ subspace as $\mathcal{H}^{\text{matt}}_{\text{cl}_0}$. The wavefunction $\psi(b)$ then takes values in the fixed, b-independent Hilbert space $\mathcal{H}^{\text{matt}}_{\text{cl}_0}$. So $\mathcal{H}_{\text{cl}} = \mathcal{H}^{\text{matt}}_{\text{cl}_0} \otimes L^2(\mathbb{R}_+)$, where $\mathbb{R}_+$ is the half-line $b > 0$.

Now let us discuss what should be the bulk Hilbert space of JT gravity in a world with two asymptotic boundaries and with wormholes included in the dynamics. A spacetime with two asymptotic boundaries must always have precisely one open component, that is, one component that is noncompact in space (open and closed universes are both noncompact in time). The Hilbert space obtained by quantizing JT gravity plus matter in a (connected) open universe was described in section 2.3. Up to this point, we have denoted that Hilbert space simply as $\mathcal{H}$, but now that we are including closed universes, it will be helpful to be more precise and write $\mathcal{H}_{\text{op}}$ for the open universe Hilbert space.

Once we include wormholes, any number of closed universes can be created and annihilated, so a bulk description of the Hilbert space (in a spacetime with one open component) will include one factor of $\mathcal{H}_{\rm op}$ and any number of factors of $\mathcal{H}_{\rm cl}$. We do, however, have to take into account Bose symmetry among the closed universes. Bose symmetry means that the Hilbert space for a closed universe with k components is not $\mathcal{H}_{\rm cl}^{\otimes k}$, but its symmetric part, often denoted $\mathrm{Sym}^k\mathcal{H}_{\rm cl}$. Thus the full bulk Hilbert space with one open component and any number of closed components is $\mathcal{H}_{\rm op}\otimes(\mathbb{C}\oplus\mathcal{H}_{\rm cl}\oplus\mathrm{Sym}^2\mathcal{H}_{\rm cl}\oplus\cdots)$. A common abbreviation is to write $\mathrm{Sym}^*\mathcal{H}_{\rm cl}=\mathbb{C}\oplus\mathcal{H}_{\rm cl}\oplus\mathrm{Sym}^2\mathcal{H}_{\rm cl}\oplus\cdots$, so finally the bulk Hilbert space for the case of one open component is

$$\mathcal{H}_{\rm bulk}=\mathcal{H}_{\rm op}\otimes\mathrm{Sym}^*\mathcal{H}_{\rm cl}. \tag{5.3}$$

Since we will study wormhole dynamics with the help of Euclidean path integrals, we also want to consider the Euclidean analog of the closed universe U_b. Setting $\tau_E=\mathrm{i}\tau$, the big bang/big crunch spacetime U_b is converted to a complete spacetime of Euclidean signature:

$$\mathrm{d}s^2=\mathrm{d}\tau_E^2+\cosh^2\tau_E\mathrm{d}\phi^2. \tag{5.4}$$

The curve $\tau=0$ in Lorentz signature, or $\tau_E=0$ in Euclidean signature, is a closed geodesic γ of length b. In Lorentz signature, this geodesic has maximal length in its homotopy class, but in Euclidean signature it has minimal length.

In general, if M is any Euclidean signature spacetime with $R=-2$ and $\gamma\subset M$ is a simple (non-self-intersecting) closed geodesic of length b, then γ is always locally length-minimizing; moreover, near γ, M is precisely isometric to U_b. We can regard b as a measure of the size of the wormhole. As remarked in section 5.1, in JT gravity coupled to matter, wormhole contributions actually diverge for $b\to 0$, though we will proceed formally and not worry about this.

5.3 The Hilbert Space From A Boundary Point of View

From a boundary point of view, we can repeat many statements from section 3.3, but now including wormhole corrections.

Thus, if S is a string, we now define $\mathrm{Tr}\,\mathsf{S}$ by a path integral on a two-manifold M that has a single asymptotic boundary labeled by S (with its ends joined together), as in fig. 11(a). This is precisely analogous to fig. 1, except that since we want to include wormhole corrections, we no longer insist that M should be a disc; rather in defining $\mathrm{Tr}\,\mathsf{S}$, we sum over all isomorphism classes of hyperbolic two-manifold of any genus with a single boundary component.

Similarly, we can define an inner product on the space $\mathcal{A}_0$ spanned by the strings in the familiar way:

$$\langle\mathsf{S},\mathsf{T}\rangle=\mathrm{Tr}\,\mathsf{S}^\dagger\mathsf{T}. \tag{5.5}$$

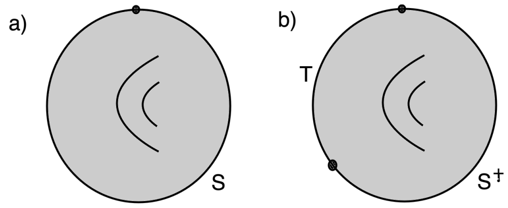

Figure 11. (a) A disc with a handle attached, representing a genus 1 contribution to $\mathrm{Tr}\,\mathsf{S}$ for some string S. The boundary is labeled by the string S, with its ends glued together to make a circle. (b) A similar procedure to compute $\langle \mathsf{S}, \mathsf{T}\rangle = \mathrm{Tr}\,\mathsf{S}^\dagger \mathsf{T}$.

Concretely, this matrix element is computed on an oriented two-manifold M with a single asymptotic boundary that is labeled by $\mathsf{S}^\dagger\mathsf{T}$ and otherwise with any topology (fig. 11(b)). We will show in section 5.5 that this inner product is positive semi-definite.[22]

The procedure to construct a Hilbert space is the same as before. We define formally for every string S a state Ψ_S, and we define the inner products of these states by $\langle \Psi_{\mathsf{S}_1}, \Psi_{\mathsf{S}_2}\rangle = \langle \mathsf{S}_1, \mathsf{S}_2\rangle$. Dividing by null vectors and taking a Hilbert space completion, we get a Hilbert space that now we call the boundary Hilbert space $\mathcal{H}_{\rm bdry}$. Its relation to the bulk Hilbert space $\mathcal{H}_{\rm bulk}$ is more subtle than was the case in section 3.3 and will be the subject of section 5.4.

As before, strings can act on $\mathcal{H}_{\rm bdry}$ by $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}}$. This gives an action of $\mathcal{A}_0$ on the Hilbert space $\mathcal{H}_{\rm bdry}$. We will explain in section 5.5 that if Ψ_S or Ψ_T is null, then $\mathsf{S}\Psi_\mathsf{T} = \Psi_{\mathsf{ST}}$ is also null. Hence it is possible to take the quotient of $\mathcal{A}_0$ by null vectors to get an algebra $\mathcal{A}_1$. Taking a completion of $\mathcal{A}_1$, we get a von Neumann algebra $\mathcal{A}$ of boundary observables that act on $\mathcal{H}_{\rm bdry}$, acting on a string on the left. Its commutant $\mathcal{A}'$ the opposite algebra $\mathcal{A}^{\rm op}$, acting on strings on the right.

The trace and therefore also the inner products that were used in this construction receive wormhole corrections, so they are not the same as they were in the absence of wormholes in section 3.3. However, in the presence of appropriate matter, the algebra $\mathcal{A}$ is a factor of Type II_∞ just as before. Wormhole corrections cannot bring a center into being, so the center is trivial if it is trivial in the absence of wormholes. Having a trace that is not defined for all elements of the algebra, $\mathcal{A}$ must then be of Type II_∞ or Type I_∞. It is not of Type I_∞, since order by order in the wormhole expansion we are not solving the black hole information problem.

5.4 Relating the Boundary and the Bulk

In section 2, we constructed a bulk Hilbert space $\mathcal{H}_{\rm bulk}$ that includes closed universes. In section 5.3, we defined a boundary Hilbert space $\mathcal{H}_{\rm bdry}$ and for every string S, a corresponding vector $\Psi_\mathsf{S} \in \mathcal{H}_{\rm bdry}$. An inner product $\langle \Psi_\mathsf{S}, \Psi_\mathsf{T}\rangle$ is computed by a path integral on

[22]This statement is nontrivial only because the inner product defined via disc amplitudes in section 3.3 is positive semi-definite rather than positive-definite. A strictly positive inner product in the absence of wormholes would automatically remain positive order by order in the wormhole expansion. The nontrivial question is whether vectors that are null vectors in leading order can gain negative norm due to wormhole corrections. We will see that this does not occur.

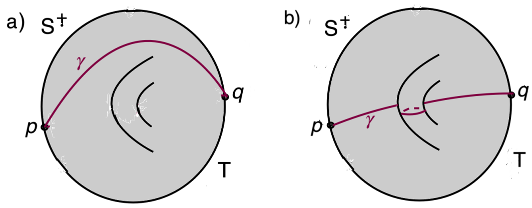

Figure 12. A contribution to $\mathrm{Tr}\,\mathsf{S}^\dagger\mathsf{T}$ from a genus one spacetime M. The boundary segments labeled by $\mathsf{S}^\dagger$ and by T are separated by points p, q. Two examples are sketched of a separating geodesic cut γ from p to q. In (a), γ is connected and consists of a geodesic from p to q. In (b), γ is not connected and is the union of a geodesic from p to q and a closed geodesic "inside the wormhole." In each case, γ is separating in the sense that removing it divides M into disconnected components "above" and "below" γ. In (b), this would not be so if we omit from γ the wormhole component.

an oriented two-manifold M of any topology with an asymptotic boundary circle labeled by $\mathsf{S}^\dagger\mathsf{T}$ and any "filling" of M in the interior. We want to find a map

$$\mathcal{W}:\mathcal{H}_{\rm bdry}\to\mathcal{H}_{\rm bulk} \tag{5.6}$$

that preserves inner products, in the sense that

$$\langle\Psi_{\mathsf{S}},\Psi_{\mathsf{T}}\rangle=\langle\mathcal{W}(\Psi_{\mathsf{S}}),\mathcal{W}(\Psi_{\mathsf{T}})\rangle, \tag{5.7}$$

where the inner product on the left is in $\mathcal{H}_{\rm bdry}$ and the one on the right is in $\mathcal{H}_{\rm bulk}$. The adjoint of $\mathcal{W}$ is a bulk to boundary map

$$\mathcal{V}:\mathcal{H}_{\rm bulk}\to\mathcal{H}_{\rm bdry}. \tag{5.8}$$

Since the inner product on $\mathcal{H}_{\rm bulk}$ is manifestly positive, the existence of the map $\mathcal{W}$ implies that, as was asserted in section 5.3, the inner product on states Ψ_{S} is positive semi-definite, so that after dividing by null vectors, the inner product on $\mathcal{H}_{\rm bdry}$ is positive-definite.

To find the map $\mathcal{W}$, we will generalize the procedure of section 3.3 to allow for the presence of wormholes. We start with the path integral that defines the inner product $\langle\Psi_{\mathsf{S}},\Psi_{\mathsf{T}}\rangle$ of states associated with strings. This is a path integral on a spacetime M whose boundary is a circle made up of segments labeled by $\mathsf{S}^\dagger$ and by T. The segments meet at boundary points p,q. Previously (fig. 2(b)), M was assumed to be a hyperbolic disc, and therefore there was a unique geodesic $\gamma\subset M$ joining p and q. This divided M into portions M_- "below" γ and M_+ "above" γ. (In what follows, we include γ itself in both M_- and M_+ and thus we define M_- and M_+ to be closed.) The path integral on M_- gives a description of a ket, the path integral on M_+ gives a description of a bra, and the sum over fields on γ computes the inner product of these two states. This is how, in the absence of wormholes, we defined a map from boundary states to bulk states that preserved inner products. In the absence of wormholes, this map was an isomorphism so we did not distinguish the boundary and bulk Hilbert spaces.

This construction needs some modification when wormholes are included, because although γ is unique when M is a disc, it is otherwise far from unique. In general, when

M has higher genus, there are infinitely many geodesics in M connecting the boundary points p and q. If we simply sum over all possible γ's, we will get an infinite overcounting. Instead of such a simple sum, we will stipulate that we pick γ to have minimal renormalized length among all geodesic cuts from p to q. By a geodesic cut from p to q, we mean a one-dimensional submanifold $\gamma \subset M$ that satisfies the geodesic equation, has asymptotic ends at the points p, q, and has the property that if we "cut" M along γ, it divides into an "upper" piece M_+ (containing in fig. 12 the part of ∂M labeled by $\mathsf{S}^\dagger$) and a "lower" piece M_- (containing the part labeled by T). As discussed in more detail shortly, we do *not* require that γ be connected. The minimal geodesic cut is unique except on a set of measure zero in the moduli space of hyperbolic metrics on M; such a set of measure zero is not important in the analysis of inner products between states in Hilbert space.

The choice of the minimal geodesic cut requires some discussion. First of all, this choice is computationally difficult in the sense that in general it is difficult to find the minimal geodesic cut. That is one reason, but probably far from being the main reason, that the boundary to bulk map $\mathcal{W}$ and its adjoint, the bulk to boundary map $\mathcal{V} = \mathcal{W}^\dagger$, are computationally difficult. These maps are relatively simple to describe (modulo the difficulty in finding minimal geodesic cuts) for states that have a simple Euclidean description, but for other states, $\mathcal{V}$ and $\mathcal{W}$ are probably very difficult to describe explicitly. For example, acting on a state with a simple Euclidean construction, real time evolution by the boundary Hamiltonians H_L and/or H_R probably produces states on which an explicit description of the maps $\mathcal{W}$ and $\mathcal{V}$ is very complicated. This will become apparent when we describe how to define H_L and H_R as operators on $\mathcal{H}_{\text{bulk}}$ (see the end of section 5.6).

A second point is that as the moduli of M are changed, the minimal geodesic cut γ generically evolves smoothly but will sometimes jump discontinuously. We are not sure what to say about this. Such jumps are possibly inevitable if one aims to give a Hamiltonian description, with continuous time evolution, of a theory in which spacetime is modeled as a smooth manifold, so that the distinction between different topologies is sharp. We use the minimal geodesic cut as a sort of gauge choice for the bulk state. Although relying on the minimal geodesic cut will probably seem unnatural to many readers, with its aid we will obtain some nice results that appear hard to obtain otherwise. With the help of the minimal geodesic cut, we can describe explicitly the map $\mathcal{W}$ from states defined by boundary data to bulk states and prove that it preserves inner products. This also makes it possible to complete the definition of the boundary Hilbert space $\mathcal{H}_{\text{bdry}}$. And the minimal geodesic cut will be a key tool in proving that any pure or mixed state on $\mathcal{H}_{\text{bulk}}$ is equivalent, for a boundary observer, to some pure state in $\mathcal{H}_{\text{bdry}}$. So the minimal geodesic cut is useful, but perhaps there is another route to the same results.

Once we decide to base the definition of the boundary to bulk map on minimal geodesic cuts, there is still a choice to make, as illustrated in fig. 12:

(1) We could stipulate that γ should be connected and thus should be simply a geodesic from p to q. We will call this the restricted version of the proposal. In this case, the cut reveals a single open universe and no closed ones. Therefore, with this proposal, the boundary to bulk map $\mathcal{W}$ really maps $\mathcal{H}_{\text{bdry}}$ to the original open universe Hilbert space $\mathcal{H}_{\text{op}} \subset \mathcal{H}_{\text{bulk}}$. However, we will see that this version of the proposal does not work.

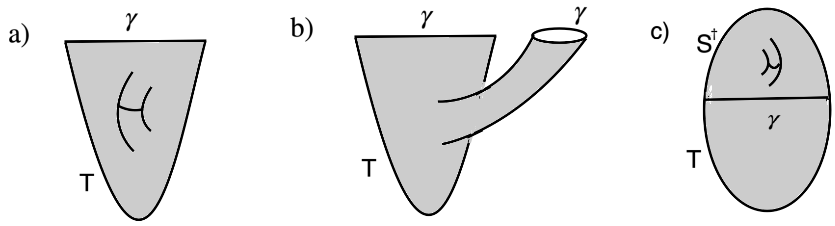

Figure 13. Illustrated here is the procedure to calculate the bulk state $\mathcal{W}(\Psi_{\mathsf{T}})$ for a string T. The spacetime M_- has an asymptotic boundary labeled by the string T as well as a minimal geodesic boundary γ. There are two possible cases. In the restricted proposal, M_- may have wormholes but γ is connected, as sketched in (a), and $\mathcal{W}(\Psi_{\mathsf{T}})$ is an element of the open universe Hilbert space $\mathcal{H}_{\text{op}}$. In the natural proposal, γ is allowed to be disconnected, as in (b), and $\mathcal{W}(\Psi_{\mathsf{T}})$ is valued in $\mathcal{H}_{\text{bulk}}$ but not in $\mathcal{H}_{\text{op}}$. As shown in (c), to compute an inner product $\langle\mathcal{W}(\Psi_{\mathsf{S}})|\mathcal{W}(\Psi_{\mathsf{T}})\rangle$, we glue together a bra and ket $|\mathcal{W}(\Psi_T)\rangle$ and $\langle\mathcal{W}(\Psi_{\mathsf{S}})|$ defined by this procedure and perform a path integral. Sketched is an example with one wormhole and a connected minimal geodesic cut γ. In case $\mathsf{S}=\mathsf{T}$, the resulting path integral is nonnegative, and vanishes if and only if $\mathcal{W}(\Psi_{\mathsf{T}})=0$, because for any values of the fields on γ, the path integral on the region above γ is the complex conjugate of the path integral on the region below γ.

(2) In the alternative that works, there is no condition for the geodesic cut γ to be connected. We do require that γ is embedded in M. Then γ consists of a simple (non-self-intersecting) geodesic γ_0 from p to q along with disjoint simple closed geodesics γ_α, $\alpha=1,\cdots,n$. In this case, when we "cut" along γ, we reveal an open universe and n closed universes. So with this definition $\mathcal{W}$ really maps $\mathcal{H}_{\text{bdry}}$ to $\mathcal{H}_{\text{bulk}}$, not just to the open universe subspace $\mathcal{H}_{\text{op}}$. We will call this the natural version of the proposal, since once wormholes are included, it seems unnatural to exclude closed universe states.

In either version of the proposal, one has to explain the rule for describing $\mathcal{W}(\Psi_{\mathsf{T}})$ as a state on γ. We expect to compute $|\mathcal{W}(\Psi_{\mathsf{T}})\rangle$ by a path integral over two-manifolds M_- that have an asymptotic boundary segment labeled by T and a geodesic boundary γ. The path integral on M_- as a function of the fields on γ will compute the desired state. In the restricted proposal, γ is required to be connected (fig. 13(a)), and in the natural version, γ can have disconnected components (fig. 13(b)). The bra $\langle\mathcal{W}(\Psi_{\mathsf{S}})|$ will be computed similarly by a path integral over a two-manifold M_+ also with γ as a geodesic boundary, and then to compute the inner product $\langle\mathcal{W}(\Psi_{\mathsf{S}})|\mathcal{W}(\Psi_{\mathsf{T}})\rangle$, we glue M_- and M_+ together along γ to make a two-manifold M, as in fig. 13. The inner product $\langle\mathcal{W}(\Psi_{\mathsf{S}})|\mathcal{W}(\Psi_{\mathsf{T}})\rangle$ defined by this procedure (fig. 13(c)) will hopefully coincide with the path integral that we would compute on M, with asymptotic boundary conditions set by S and T.

In order for this to be true, in either version of the proposal, we need a further condition on γ to ensure that once we glue M_- and M_+ together along γ to make M, γ will be uniquely determined (at least generically) just from the geometry of M. If and only if this is so, pairs M,γ will be classified (generically) by the same data that would classify M alone, and hence the path integral evaluated with the cutting procedure will coincide with the path integral that we would have defined on M if we had never introduced γ or the decomposition of M as $M_+\cup M_-$. Our strategy to ensure that γ is uniquely determined

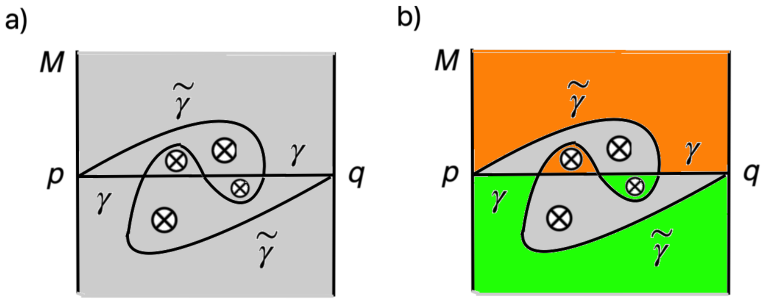

Figure 14. (a) Sketched is a portion of a two-manifold M with two geodesic cuts γ and $\widetilde{\gamma}$ between boundary points p and q; $\widetilde{\gamma}$ is partly "above" and partly "below" γ. The symbols $\otimes$ represent unspecified topological complications (such as an attached genus g surface). The figure is drawn so that γ is a horizontal straight line on the page that looks like a geodesic. In the presence of the indicated wormholes, $\widetilde{\gamma}$ might be a geodesic as well. The relation between γ and $\widetilde{\gamma}$ is symmetric; by a diffeomorphism of M, one could make $\widetilde{\gamma}$ look like a straight line on the page and make γ look like a wiggly curve. So either one could have smaller renormalized length. (b) γ_- is the boundary of the green region; γ_+ is the boundary of the orange region. They are not connected.

(generically) will be to arrange so that γ is a minimal geodesic cut from p to q in M. For this to have a chance of being true, we have to at least require that γ is minimal in M_-, meaning that there is no cut from p to q in M_- (or no connected cut in the restricted version of the proposal) that has a renormalized length less than γ. Here in the case of a manifold M_- with boundary, we allow a geodesic cut to be contained partly or entirely in the boundary (thus the boundary of M_- is regarded as an example of a cut, even though in this case the part of M_- "above" the cut is empty). In asking that γ should be minimal, it does not matter if we ask for γ to be minimal among all cuts or only among geodesic cuts; if there is a non-geodesic cut from p to q that is shorter than γ, then it can always be further shortened to a geodesic cut that is also shorter than γ. Similarly we require that γ is minimal in M_+. In either the revised or the natural version of the proposal, stipulating that we only integrate over metrics on M_- with the property that the boundary γ is minimal (among all cuts in the natural version of the proposal, and among all connected cuts in the restricted version) completes the definition of what we mean by the path integral on M_- that, as a function of fields on γ, is supposed to compute $|\mathcal{W}(\Psi_\mathsf{T})\rangle$. A similar restricted path integral on M_+ computes a bra of the form $\langle\mathcal{W}(\Psi_\mathsf{S})|$.

But when we glue together M_- and M_+ to make M, is γ minimal in M, or can it be replaced in M by a shorter geodesic cut $\widetilde{\gamma}$ from p to q? Since γ was minimal in M_-, there is no such $\widetilde{\gamma}$ that is contained entirely in M_-, and since γ was minimal in M_+, there is no such $\widetilde{\gamma}$ that is contained entirely in M_+. But could there be a $\widetilde{\gamma}$ that is partly in M_+ and partly in M_- (fig. 14(a))?

In the natural version of the proposal, a simple cut and paste argument shows that if γ is minimal in M_- and in M_+, then it is minimal in M. This argument does not work for the restricted version of the proposal. That is why only the natural version of the proposal is successful.

To explain the cut and paste procedure, let $\widetilde{\gamma}$ be any cut from p to q. Just as γ divides

M into a lower piece M_- and an upper piece M_+, likewise $\widetilde{\gamma}$ divides M into a lower piece $\widetilde{M}_-$ and an upper piece $\widetilde{M}_+$. Now we can define two new cuts, $\gamma_- = \gamma \cap \widetilde{M}_- \cup \widetilde{\gamma} \cap M_-$, and $\gamma_+ = \gamma \cap \widetilde{M}_+ \cup \widetilde{\gamma} \cap M_+$. In other words, γ_- consists of points in γ that are "below" (or on) $\widetilde{\gamma}$ together with points in $\widetilde{\gamma}$ that are "below" γ, while γ_+ consists of points in γ that are "above" (or on) $\widetilde{\gamma}$ and points in $\widetilde{\gamma}$ that are "above" γ. Equivalently, γ_- is the boundary of $M_- \cap \widetilde{M}_-$ and γ_+ is the boundary of $M_+ \cap \widetilde{M}_+$. This last description makes clear that γ_- and γ_+ are cuts. Note in particular that $\gamma_- \subset M_-$ and $\gamma_+ \subset M_+$. These definitions imply that $\gamma_- \cup \gamma_+ = \gamma \cup \widetilde{\gamma}$ and[23] $\gamma_- \cap \gamma_+ = \gamma \cap \widetilde{\gamma}$. Accordingly, the renormalized lengths of the four cuts satisfy

$$\ell(\gamma_+) + \ell(\gamma_-) = \ell(\gamma) + \ell(\widetilde{\gamma}). \tag{5.9}$$

In the natural version of the proposal, minimality of γ in M_- means that the renormalized length of γ_- is no less than that of γ:

$$\ell(\gamma_-) \geq \ell(\gamma). \tag{5.10}$$

Similarly, minimality of γ in M_+ implies in the natural version that

$$\ell(\gamma_+) \geq \ell(\gamma). \tag{5.11}$$

A linear combination of these relations gives

$$\ell(\widetilde{\gamma}) \geq \ell(\gamma), \tag{5.12}$$

showing, in the natural version of the proposal, that γ is minimal in M.

Why does this argument fail in the restricted version of the proposal? It fails because even if γ and $\widetilde{\gamma}$ are connected, γ_- and γ_+ may not be (fig. 14(b)). If γ_- or γ_+ is not connected, then in the restricted version of the proposal, we are not entitled to assume eqn. (5.10) or eqn. (5.11), so we cannot deduce eqn. (5.12). On the contrary, fig. 14 is essentially symmetrical in γ and $\widetilde{\gamma}$ up to a diffeomorphism of M, so in the restricted version of the proposal, it is entirely possible for γ to be non-minimal.

Although the boundary-to-bulk map $\mathcal{W} : \mathcal{H}_{\text{bdry}} \to \mathcal{H}_{\text{bulk}}$ is isometric and well-defined for any boundary state, there is no reason to think that its image is dense is $\mathcal{H}_{\text{bulk}}$, and hence no reason to think that the adjoint map $\mathcal{V} : \mathcal{H}_{\text{bulk}} \to \mathcal{H}_{\text{bdry}}$ is also isometric. In particular, in the limit $e^{-S} \to 0$ where the wormhole contributions vanish, the boundary path integral defined by a string S can be used to prepare arbitrary states in the Hilbert space $\mathcal{H}_{\text{op}}$ of an open geodesic, but does not enable us to create the states that contain closed universes. Intuitively, the "size" of $\mathcal{W}(\mathcal{H}_{\text{bdry}})$ is independent of e^{-S}, so we expect $\mathcal{W}(\mathcal{H}_{\text{bdry}})$ to be much "smaller" than $\mathcal{H}_{\text{bulk}}$ for all values of e^{-S}.

[23]Unless γ and $\widetilde{\gamma}$ have one or more components in common (which is possible in the natural version of the proposal if γ and $\widetilde{\gamma}$ are not connected), $\gamma \cap \widetilde{\gamma}$ is a set of measure 0, possibly a finite set. In the example of fig. 14, $\gamma \cap \widetilde{\gamma}$ consists of three points. Common components of γ and $\widetilde{\gamma}$, if there are any, are also present in γ_+ and γ_- and cancel out of all relations in the text.

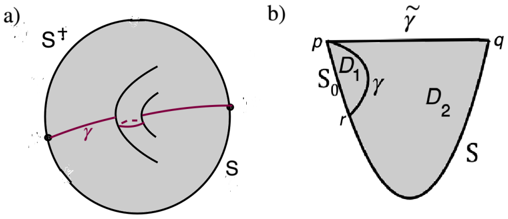

Figure 15. (a) A one wormhole contribution to $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{S}} \rangle$. γ is a minimal geodesic cut with two components. For fixed values of the fields along γ, after summing over all topologies and integrating over all moduli, the path integrals above and below γ are complex conjugates, implying that $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{S}} \rangle \geq 0$. (b) This is a repeat of fig. 4 except that wormholes may be present (not drawn) and γ and $\widetilde{\gamma}$ are now minimal geodesic cuts. If Ψ_{S_0} is null, then the path integral in the smaller region D_1 vanishes for any values of the fields along γ. This implies vanishing of the path integral in $D_0 = D_1 \cup D_2$, implying that $\Psi_{\mathsf{S}_0\mathsf{S}}$ is null.

5.5 Further Steps

At this point, restricting to the natural version of the proposal, it is fairly straightforward to imitate other arguments in section 3.3, with a few new twists because the boundary to bulk map $\mathcal{W} : \mathcal{H}_{\rm bdry} \to \mathcal{H}_{\rm bulk}$ is now not an isomorphism but an embedding in a larger Hilbert space.

First of all, as promised in section 5.3, we can now show that the inner products on states Ψ_{S}, with $\mathsf{S} \in \mathcal{A}_0$, are positive semi-definite. In the path integral that computes $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{S}} \rangle$, which is sketched in fig. 15(a), for any values of the fields on the minimal geodesic cut γ, after integrating over all moduli, the path integral on the region above the cut is equal to the complex conjugate of the path integral below the cut. This is true essentially by reflection positivity of the bulk path integral. (More precisely, it is true because orientation reversal has the effect of complex conjugating the integrand of the bulk path integral; this is the fact that underlies reflection positivity.) Hence $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{S}} \rangle \geq 0$, with vanishing only if the bulk state $\mathcal{W}(\Psi_{\mathsf{S}})$ vanishes identically as a function of the fields on γ. This enables us to define a boundary Hilbert space $\mathcal{H}_{\rm bdry}$ together with an embedding $\mathcal{W} : \mathcal{H}_{\rm bdry} \to \mathcal{H}_{\rm bulk}$.

As before, we declare $\mathsf{S} \in \mathcal{A}_0$ to be null if $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{S}} \rangle = 0$ and let $\mathcal{A}_1$ be the quotient of $\mathcal{A}_0$ by such null vectors. To know that $\mathcal{A}_1$ is an algebra and acts on $\mathcal{H}_{\rm bdry}$, we need to know that if S_0 is null, then $\mathsf{S}_0\mathsf{S}$ and $\mathsf{S}\mathsf{S}_0$ are also null. This follows by the same argument as before, with geodesics replaced by minimal geodesic cuts (fig. 15(b)). So now we can take the completion of $\mathcal{A}_1$ as an algebra acting on $\mathcal{H}_{\rm bdry}$. This completion is the algebra $\mathcal{A} = \mathcal{A}_L$ of observables on the left boundary. Acting on $\mathcal{H}_{\rm bdry}$, $\mathcal{A}_L$ has a commutant that consists of a similar algebra $\mathcal{A}' = \mathcal{A}_R$ of observables on the right boundary.

We now want to define an action of $\mathcal{A}$ on $\mathcal{H}_{\rm bulk}$. In section 3.3, this step was vacuous since $\mathcal{H}_{\rm bdry}$ and $\mathcal{H}_{\rm bulk}$ coincided. First of all, for a string T and states $\Psi, \Psi' \in \mathcal{H}_{\rm bulk}$, we define the matrix elements $\langle \Psi' | \mathsf{T} | \Psi \rangle$ by a path integral in a spacetime region M_1 schematically depicted in fig. 16(a). M_1 has an asymptotic boundary segment labeled by T, and it has past and future boundaries given by geodesic 1-manifolds γ and γ', which are not necessarily connected. Initial and final states Ψ and Ψ' are inserted on γ and γ'. Though

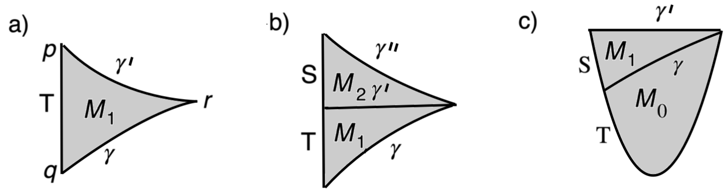

Figure 16. (a) A path integral in region M_1 can be used to compute matrix elements $\langle\Psi'|\mathsf{T}|\Psi\rangle$, where Ψ and Ψ' are bulk states inserted on γ and γ', respectively. (Wormholes and initial and final closed universes may be present and are not drawn.) (b) The path integral in $M_{12} = M_1 \cup M_2$ is used to prove that the definition in (a) gives an action of the algebra $\mathcal{A}$ of boundary observables acts on the bulk Hilbert space. (c) The path integral on $M_{01} = M_0 \cup M_1$ is used to show that the boundary to bulk map $\mathcal{W}$ commutes with the action of boundary observables. Note that this picture also implies that if $\Psi_{\mathsf{T}} = 0$ then $\Psi_{\mathsf{ST}} = 0$.

not drawn in the figure, M_1 may have wormholes and γ and γ' may have any number of disconnected components, corresponding to the possible presence of closed universes in the initial and final states. The path integral over M_1 is carried out only over hyperbolic metrics such that γ and γ' are minimal.

To show that this definition does give an action of $\mathcal{A}_1$ on $\mathcal{H}_{\mathrm{bulk}}$, we need to show that for strings S, T, we have

$$\langle\Psi''|\mathsf{ST}|\Psi\rangle = \sum_{\Psi'} \langle\Psi''|\mathsf{S}|\Psi'\rangle\langle\Psi'|\mathsf{T}|\Psi\rangle. \tag{5.13}$$

Here the three matrix elements $\langle\Psi''|\mathsf{ST}|\Psi\rangle$, $\langle\Psi''|\mathsf{S}|\Psi'\rangle$, and $\langle\Psi'|\mathsf{T}|\Psi\rangle$ are all supposed to be computed by the recipe just stated, and the sum over Ψ' runs over an orthonormal basis of $\mathcal{H}_{\mathrm{bulk}}$. The picture that corresponds to this identity is shown in fig. 16(b). In this picture, the spacetime M_{12} has an asymptotic boundary labeled by S and T, geodesic boundaries γ and γ'', on which initial and final states Ψ and Ψ'' are inserted, and an internal geodesic cut γ'. T, γ, and γ' bound a "lower" piece M_1 of M_{12}, while S, γ', and γ'' bound an "upper" piece M_2. M_{12} is built by gluing together M_1 and M_2 along their common boundary γ'. If γ and γ' are minimal in M_1, and γ' and γ'' are minimal in M_2, then the path integral on M_{12} computes the right hand side of eqn. (5.13), with the sum over intermediate states Ψ' coming from the sum over fields on γ'. On the other hand, if γ, γ', and γ'' are all minimal in M_{12}, then the same path integral computes the left hand side of eqn. (5.13). Here minimality of γ' means that generically it is uniquely determined by the geometry of M_{12}, so including it in the definition of the path integral has no effect and it can be forgotten; minimality of γ and γ'' in M_{12} is the condition that defines the path integral on M_{12} that computes $\langle\Psi''|\mathsf{ST}|\Psi\rangle$. So to complete the proof of eqn.(5.13), we just need to know that if γ and γ' are minimal in M_1, and γ' and γ'' are minimal in M_2, then all three of them are minimal in M_{12}. This follows by the same cut and paste argument as in section 5.4.

At this point, it is natural to ask if the boundary to bulk map $\mathcal{W} : \mathcal{H}_{\mathrm{bdry}} \to \mathcal{H}_{\mathrm{bulk}}$ is compatible with the action of the algebra $\mathcal{A}_1$ on $\mathcal{H}_{\mathrm{bdry}}$ and on $\mathcal{H}_{\mathrm{bulk}}$ in the sense that for a string S and for $\Psi \in \mathcal{H}_{\mathrm{bdry}}$, one has $\mathsf{S}\mathcal{W}(\Psi) = \mathcal{W}(\mathsf{S}\Psi)$. It suffices to check this for the case that $\Psi = \Psi_{\mathsf{T}}$ for some string T, since states of that form are dense. Thus we need to

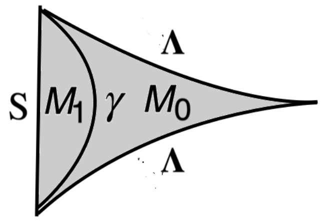

Figure 17. A picture that is used to show that, from the point of view of a boundary observer, any bulk state is equivalent to a possibly mixed state on $\mathcal{H}_{\rm bdry}$.

verify that

$$\mathsf{S}\mathcal{W}(\Psi_{\mathsf{T}}) = \mathcal{W}(\Psi_{\mathsf{ST}}). \tag{5.14}$$

The relevant picture is fig. 16(c), where if the relevant cuts are minimal, then (i) the path integral in region M_0 computes $\mathcal{W}(\Psi_{\mathsf{T}})$, (ii) the path integral in region M_0 computes the action of S on this state, and (iii) the path integral in $M_{01} = M_0 \cup M_1$ computes $\mathcal{W}(\Psi_{\mathsf{ST}})$. For statement (i), γ must be minimal in M_0, for statement (ii), γ and γ' must be minimal in M_1, and for statement (iii), γ and γ' must be minimal in M_{01}. (For statement (iii), we reason as in the last paragraph: γ being minimal in M_{01} means that it is uniquely determined generically and plays no role, and γ' being minimal and the hyperbolic metric of M_{01} being otherwise arbitrary ensures that the path integral in M_{01} computes $\mathcal{W}(\Psi_{\mathsf{ST}})$.) A cut and paste argument as before shows that if γ is minimal in M_0 and γ and γ' are minimal in M_1, then γ and γ' are minimal in M_{01}. So eqn. (5.14) is valid.

This argument shows that the algebra $\mathcal{A}_1$ acts on $\mathcal{H}_{\rm bdry}$ and $\mathcal{H}_{\rm bulk}$, and that $\mathcal{W}: \mathcal{H}_{\rm bdry} \to \mathcal{H}_{\rm bulk}$ maps the action on $\mathcal{H}_{\rm bdry}$ to the action on $\mathcal{H}_{\rm bulk}$. We can complete $\mathcal{A}_1$ acting on $\mathcal{H}_{\rm bulk}$ to a von Neumann algebra, and we want to know that this is the same von Neumann algebra $\mathcal{A} = \mathcal{A}_L$ that we get if we complete $\mathcal{A}_1$ acting on $\mathcal{H}_{\rm bdry}$. This statement means for any sequence $\mathsf{S}_1, \mathsf{S}_2, \cdots \in \mathcal{A}_1$, the sequence $\lim_{n\to\infty}\langle\Psi|\mathsf{S}_n|\Psi\rangle$ converges for all $\Psi \in \mathcal{H}_{\rm bulk}$ if and only if it converges for all $\Psi \in \mathcal{H}_{\rm bdry}$. The "only if" statement is trivial since $\mathcal{H}_{\rm bdry}$ is isomorphic via $\mathcal{W}$ to a subspace of $\mathcal{H}_{\rm bulk}$. The "if" statement follows from something superficially stronger but actually equivalent that we will prove presently: for every $\Lambda \in \mathcal{H}_{\rm bulk}$, there is $\chi \in \mathcal{H}_{\rm bdry}$ such that $\langle\Lambda|\mathsf{S}|\Lambda\rangle = \langle\chi|\mathsf{S}|\chi\rangle$ for all $\mathsf{S} \in \mathcal{A}_1$. Physically, this statement means that every $\Lambda \in \mathcal{H}_{\rm bulk}$ is indistinguishable, from the point of view of a boundary observer, from some $\chi \in \mathcal{H}_{\rm bdry}$.

A tempting but insufficient argument would go as follows. If $\Lambda \in \mathcal{H}_{\rm bulk}$ is any bulk state, the function $\mathsf{S} \to \langle\Lambda|\mathsf{S}|\Lambda\rangle$ is a linear functional on the algebra $\mathcal{A}$ that is non-negative (meaning that it is non-negative if $\mathsf{S} = \mathsf{T}^\dagger\mathsf{T}$ for some T), and therefore, as $\mathcal{A}$ is of Type II_∞, it is $\mathrm{Tr}\,\mathsf{S}\rho$ for some ρ. Here ρ may be an element of $\mathcal{A}$, but more generally is "affiliated" to $\mathcal{A}$ (meaning that bounded functions of ρ belong to $\mathcal{A}$ and in particular that ρ can be arbitrarily well approximated for many purposes by elements of $\mathcal{A}$). ρ can then also be replaced by a pure state $|\chi\rangle$, as we explain later. The trouble with this argument is that the function $\langle\Lambda|\mathsf{S}|\Lambda\rangle$, for $\Lambda \in \mathcal{H}_{\rm bulk}$, is initially defined for S in the algebra $\mathcal{A}_1$ of linear combinations of strings modulo null vectors. To know that this linear function is $\mathrm{Tr}\,\mathsf{S}\rho$ for some ρ in (or affilliated to) $\mathcal{A}$, we need to know that it extends continuously over the completion $\mathcal{A} = \mathcal{A}_L$ of $\mathcal{A}_1$, or equivalently, that the von Neumann algebra completion of $\mathcal{A}_1$ acting on $\mathcal{H}_{\rm bulk}$ is the same as[24] $\mathcal{A}$, which was defined as the completion of $\mathcal{A}_1$ acting on $\mathcal{H}_{\rm bdry}$. But this is what we are trying to prove.

In fact, the existence of a suitable ρ affiliated to $\mathcal{A}$ (and thus the equivalence of the two completions of $\mathcal{A}_1$) can be deduced as follows. In fig. 17, $M_{01} = M_0 \cup M_1$ has an asymptotic boundary labeled by S as well as past and future geodesic boundaries labeled by Λ that are assumed to be minimal. The path integral on M_{01}, with Λ inserted as an initial and final state on the geodesic boundaries, will compute $\langle\Lambda|\mathsf{S}|\Lambda\rangle$. But we can also choose a minimal geodesic cut γ connecting the two ends of S, as shown. Then the path integral on M_0 computes a state $\chi \in \mathcal{H}_{\rm bulk}$, and the path integral on M_1 computes the inner product $\langle\chi|\mathcal{W}(\Psi_{\mathsf{S}})\rangle$. So $\langle\Lambda|\mathsf{S}|\Lambda\rangle = \langle\chi|\mathcal{W}(\Psi_{\mathsf{S}})\rangle$. Here *a priori* χ is a general bulk state, not in the image of $\mathcal{W} : \mathcal{H}_{\rm bdry} \to \mathcal{H}_{\rm bulk}$. However, since $\mathcal{W}(\Psi_{\mathsf{S}})$ is in the image of $\mathcal{W}$, without changing the inner product $\langle\chi|\mathcal{W}(\Psi_{\mathsf{S}})\rangle$ we can replace χ by its orthogonal projection to the image of $\mathcal{W}$. Then, since $\mathcal{W}$ is invertible when restricted to this image, there is a unique $\zeta \in \mathcal{H}_{\rm bdry}$ with $\mathcal{W}(\zeta) = \chi$. In fact, $\zeta = \mathcal{V}(\chi)$, where $\mathcal{V} : \mathcal{H}_{\rm bulk} \to \mathcal{H}_{\rm bdry}$ is the adjoint of $\mathcal{W}$. So $\langle\Lambda|\mathsf{S}|\Lambda\rangle = \langle\zeta, \Psi_{\mathsf{S}}\rangle$, where now the inner product is between two states in $\mathcal{H}_{\rm bdry}$. Finally, we recall that $\mathcal{H}_{\rm bdry}$ has a dense set of states Ψ_ρ, $\rho \in \mathcal{A}$, so $\zeta = \Psi_\rho$ where ρ is either an element of $\mathcal{A}$ or in general an operator affiliated to $\mathcal{A}$. Hence $\langle\zeta, \Psi_{\mathsf{S}}\rangle = \langle\Psi_\rho, \mathsf{S}\rangle = \mathrm{Tr}\,\rho\mathsf{S}$. Putting all this together, we learn that

$$\langle\Lambda|\mathsf{S}|\Lambda\rangle = \mathrm{Tr}\,\rho\mathsf{S}. \tag{5.15}$$

ρ is a non-negative self-adjoint operator of trace 1 or in other words a density matrix, since the functional $\langle\Lambda|\mathsf{S}|\Lambda\rangle$ is nonnegative and (if Λ is a unit vector) equals 1 if $\mathsf{S} = 1$.

A general mixed state on $\mathcal{H}_{\rm bulk}$ is $\rho_{\rm bulk} = \sum_i p_i|\Lambda_i\rangle\langle\Lambda_i|$, where Λ_i are orthonormal pure states in $\mathcal{H}_{\rm bulk}$ and $p_i > 0$, $\sum_i p_i = 1$. As just explained, there are density matrices ρ_i affiliated to $\mathcal{A}$ such that $\langle\Lambda_i|\mathsf{S}|\Lambda_i\rangle = \mathrm{Tr}\,\mathsf{S}\rho_i$ for all S. So if $\rho_{\rm bdry} = \sum_i p_i\rho_i$ then

$$\mathrm{Tr}\,\mathsf{S}\rho_{\rm bulk} = \mathrm{Tr}\,\mathsf{S}\rho_{\rm bdry}. \tag{5.16}$$

On the left, S is an operator on $\mathcal{H}_{\rm bulk}$ and the trace is the natural trace of an operator that acts on $\mathcal{H}_{\rm bulk}$. On the right, S and ρ are elements of the Type II_∞ algebra $\mathcal{A}$ and Tr is the trace of this algebra. Eqn. (5.16) expresses the fact that every pure or mixed state on $\mathcal{H}_{\rm bulk}$ can be described, from the point of view of a boundary observer, by a density matrix $\rho_{\rm bdry}$ associated to $\mathcal{A}$. But actually, any such density matrix can be purified by a pure state in $\mathcal{H}_{\rm bdry}$. For this, let $\sigma = \rho_{\rm bdry}^{1/2}$. Since $\mathrm{Tr}\,\sigma^\dagger\sigma = \mathrm{Tr}\,\rho_{\rm bdry} = 1$, the condition of eqn. (3.25) is satisfied, and there is a state $|\sigma\rangle \in \mathcal{H}_{\rm bdry}$ satisfying

$$\langle\sigma|\mathsf{S}|\sigma\rangle = \mathrm{Tr}\,\mathsf{S}\sigma\sigma^\dagger = \mathrm{Tr}\,\mathsf{S}\rho_{\rm bdry} \tag{5.17}$$

[24] If the completion of $\mathcal{A}_1$ acting on $\mathcal{H}_{\rm bulk}$ were some other von Neumann algebra $\widetilde{\mathcal{A}} \neq \mathcal{A}$, then for a bulk state Λ, the density matrix ρ satisfying $\langle\Lambda|\mathsf{S}|\Lambda\rangle = \mathrm{Tr}\,\mathsf{S}\rho$ would be affiliated to $\widetilde{\mathcal{A}}$, not to $\mathcal{A}$.

for all S. Thus, in fact, to a boundary observer, every pure or mixed state $\rho_{\rm bulk}$ on the bulk Hilbert space $\mathcal{H}_{\rm bulk}$ is indistinguishable from some pure state $|\sigma\rangle$ in the much smaller Hilbert space $\mathcal{H}_{\rm bdry}$. This pure state is unique only up to the action of a unitary operator in the commutant $\mathcal{A}' = \mathcal{A}_R$ of $\mathcal{A} = \mathcal{A}_L$.

5.6 Some Final Remarks

We defined $\mathcal{H}_{\rm bdry}$ starting with open universe observables that we called "strings." A string corresponds to a piece of the asymptotic boundary of spacetime, topologically a closed interval, labeled by operator insertions that correspond to boundary observables. But $\mathcal{H}_{\rm bulk}$ is much bigger than $\mathcal{H}_{\rm bdry}$. How can states that are in $\mathcal{H}_{\rm bulk}$ but not in $\mathcal{H}_{\rm bdry}$ be accessed? One answer, in the spirit of [22], is that we could generalize the construction of section 3.3 to include "closed strings" as well as the "open strings" that we have considered so far. A closed string here just means an asymptotic boundary of spacetime that is topologically a circle. For the closed string, one can take the same boundary conditions, labeled by a pair b, Λ, that we assumed in section 4,[25] where we tried to use closed strings to define operators $\mathcal{O}_{b,\Lambda}$ on $\mathcal{H}_{\rm bdry}$. The construction would be similar to that of section 4, except that now (as in [22]), we would specify whether a given closed asymptotic boundary is creating part of the initial state or part of the final state. In section 4, there was no reason to make this distinction.

In more detail, we would proceed as follows, Let $\widehat{\mathsf{S}}$ be a not necessarily connected string consisting of a single "open string" and any number of closed strings. For each $\widehat{\mathsf{S}}$, formally define a state $\Psi_{\widehat{\mathsf{S}}}$, with inner products $\langle \Psi_{\widehat{\mathsf{T}}}, \Psi_{\widehat{\mathsf{S}}} \rangle$ defined as in section 3.3 by a path integral on a spacetime whose asymptotic boundary is built by gluing $\widehat{\mathsf{S}}$ onto the adjoint of $\widehat{\mathsf{T}}$. (The adjoint operation is the same as before for open strings, and is CPT for closed strings.) If the inner products $\langle \Psi_{\widehat{\mathsf{T}}}, \Psi_{\widehat{\mathsf{S}}} \rangle$ are positive semi-definite, then upon dividing by null vectors and taking a Hilbert space completion, one arrives at what we will call the Marolf-Maxfield Hilbert space $\mathcal{H}_{\rm MM}$, since this construction for the closed strings was described in [22]. We expect that the construction that we have described with the minimal geodesic cuts, extended to this more general situation in a natural way, will show that the inner products $\langle \Psi_{\widehat{\mathsf{T}}}, \Psi_{\widehat{\mathsf{S}}} \rangle$ are positive semi-definite and establish an isomorphism between $\mathcal{H}_{\rm MM}$ and what we have called $\mathcal{H}_{\rm bulk}$.

Of course, this reformulation of what one wants to do with asymptotic closed boundaries does not, by itself, eliminate the problem we had in section 4 with the divergence that results from the negative Casimir energy. Exactly what happens in a better theory that resolves this divergence remains to be understood.

Though every state in $\mathcal{H}_{\rm bulk}$ is equivalent from the viewpoint of a boundary observer to some pure state in $\mathcal{H}_{\rm bdry}$, there is no natural way to exhibit this equivalence by a linear map from $\mathcal{H}_{\rm bulk}$ to $\mathcal{H}_{\rm bdry}$. The only natural map that we have found between these spaces is $\mathcal{V} : \mathcal{H}_{\rm bulk} \to \mathcal{H}_{\rm bdry}$, the adjoint of $\mathcal{W}$. However, $\mathcal{V}$ is far from being an isometry: it is an isomorphism on $\mathcal{W}(\mathcal{H}_{\rm bulk})$ and annihilates the orthocomplement of this space. Still, in the

[25] One can contemplate more general boundary conditions on asymptotic closed boundaries, but we do not expect that this would add anything, since the boundary conditions considered in section 4 suffice to create an arbitrary closed universe state.

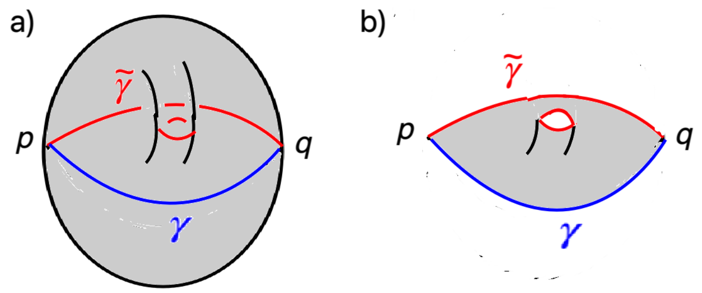

Figure 18. (a) A disc with a handle attached; shown are geodesic cuts γ (blue) and $\tilde{\gamma}$ (red) that connect boundary points p, q. In this example, γ is connected and $\tilde{\gamma}$ is not. As p is moved "upwards" along the boundary, the minimal geodesic cut can jump from γ to $\tilde{\gamma}$. (b) To reproduce this jumping, the boundary Hamiltonian H_L has a matrix element that glues the indicated surface onto an initial state defined on γ, producing a final state defined on $\tilde{\gamma}$. In this example, that matrix element involves creation of a baby universe.

spirit of [40], one may wonder whether for some purposes, after restricting to a suitable subspace of $\mathcal{H}_{\rm bulk}$, such as a subspace obtained by long enough real time evolution starting from a space of macroscopically similar black hole states, some multiple of $\mathcal{V}$ may be very hard to distinguish from an isomorphism.

We conclude with a discussion of real time evolution. Since the algebra $\mathcal{A} = \mathcal{A}_L$ of observables on the left boundary acts on the bulk Hilbert space $\mathcal{H}_{\rm bulk}$, in particular this gives an action of its generator $e^{-\beta H_L}$ on $\mathcal{H}_{\rm bulk}$. Taking logarithms, the boundary Hamiltonian H_L is an operator on $\mathcal{H}_{\rm bulk}$, and exponentiating again, we can describe the real time evolution of a bulk state by the action of $e^{-{\rm i}tH_L}$. By the same logic, we can define the evolution of a bulk state under real time evolution of the right boundary by $e^{-{\rm i}tH_R}$. Apart from topology-changing processes, H_L and H_R act very simply; they act as described in eqn. (2.39) on the open universe Hilbert space $\mathcal{H}_{\rm op}$, and they annihilate the closed universe Hilbert space $\mathcal{H}_{\rm cl}$, since the total energy of a closed universe is 0. However, this is far from the whole story; $e^{-\beta H_L}$ and similarly $e^{-\beta H_R}$ have matrix elements that describe topology-changing processes in the bulk, and therefore the same is true of H_L and H_R. For an example, see fig. 18. Because the Hamiltonian has topology-changing matrix elements, real time evolution over any substantial time interval is likely to be quite complicated, even if the starting point is a state with a simple Euclidean description.

6 Multiple Open Universes

In section 5, we studied universes with a single open component and any number of closed components. From the standpoint of General Relativity, it is certainly possible to contemplate universes with multiple open components. This generalization is nontrivial in the presence of wormholes, since different open universe components can interact by exchang-

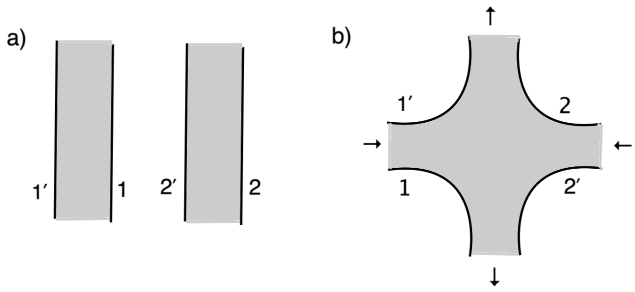

Figure 19. (a) Free propagation of an open universe of type $1'1$ and one of type $2'2$. (b) A transition from a universe with components $1'1 + 2'2$ to one with components $1'2 + 2'1$. Incoming arrows mark open universe components in the initial state, and outgoing arrows mark open universe components in the final state. Topologically, the spacetime in (a) is a disjoint union of two discs, with total Euler characteristic 2, and the spacetime in (b) is a single disc, with Euler characteristic 1. So the process in (b) is suppressed by a single power of e^{-S}. This is the lowest order "interaction" between distinct open universe components.

ing wormholes. The analysis presented so far in this article extends naturally to the case of multiple open universes, as we will now discuss.

The most significant conclusion that we will reach is that an observer with access to only one asymptotic boundary has no way to determine by any measurement how many other such boundaries there are. The reasoning that leads to this conclusion will be similar to arguments that we have seen already.

As in footnote 21, we do not assume time-reversal or reflection symmetry, so we distinguish left and right asymptotic boundaries. In the absence of time-reversal symmetry, spacetime is oriented, and its boundary is also oriented. In all pictures in this section, the orientation comes from the counterclockwise orientation of the plane.

In the absence of reflection symmetry, left and right boundaries are inequivalent; there is no Bose symmetry between them. But we also do not impose Bose symmetry between boundaries of the same type. The different left or right boundaries are considered inequivalent, since we want to analyze the operators available to an observer who has access to one specified left or right boundary.

In generalizing our previous results to a universe with multiple open components, we will not be as detailed as we have been up to this point. We just briefly describe the analogs of the main steps in section 5. In doing so, for brevity we consider the case of a universe with two open components. The generalization to any number of open components is immediate.

1) The Algebra

The algebra of observables on a particular left (or right) boundary is taken to be precisely the same algebra as in section 5 (with wormhole corrections included). Thus the algebra is defined as before by starting with strings S, T, computing inner products $\langle \Psi_{\mathsf{S}}, \Psi_{\mathsf{T}} \rangle$ from a spacetime with one asymptotic boundary and any topology, and taking a completion to get a Hilbert space $\mathcal{H}_{\text{bdry}}$ and an algebra $\mathcal{A} = \mathcal{A}_L$ that acts on it.

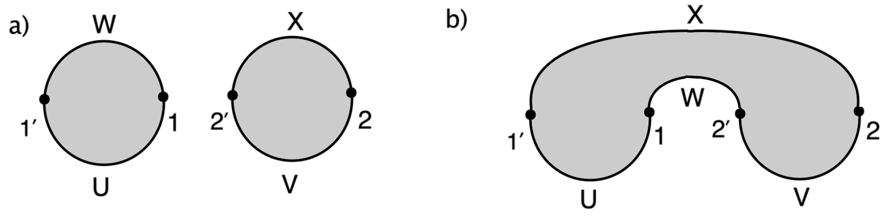

Figure 20. (a) The lowest order contribution to the inner product $\langle\Psi_{\mathsf{W}_{1'1}\times\mathsf{X}_{2'2}}|\Psi_{\mathsf{U}_{1'1}\times\mathsf{V}_{2'2}}\rangle$, between states created by string pairs both of type $1'1+2'2$. (b) The lowest order contribution to an inner product $\langle\Psi_{\mathsf{W}_{2'1}\times\mathsf{X}_{1'2}}|\Psi_{\mathsf{U}_{1'1}\times\mathsf{V}_{2'2}}\rangle$ between states created by string pairs of opposite types $1'1+2'2$ and $1'2+2'1$. Either picture can be decorated with wormholes, including wormholes that connect the two components in (a). The spacetime in (b) has Euler characteristic 1, compared to 2 in (a), so the inner product between states created by string pairs of opposite type is suppressed by one factor of e^{-S}.

2) The Bulk Hilbert Space

For the case of two open components, let us label the left boundaries as $1'$ and $2'$ and the right boundaries as 1 and 2. To make a world with these two asymptotic boundaries, we pair up the boundaries as $1'1+2'2$ or as $1'2+2'1$. Correspondingly, the Hilbert space with two open universe components (and no closed universes) is, in an obvious notation, $\mathcal{H}_{\text{op},[2]}=\mathcal{H}_{1'1}\otimes\mathcal{H}_{2'2}\oplus\mathcal{H}_{1'2}\otimes\mathcal{H}_{2'1}$. Including closed universes as before, the bulk Hilbert space with two open components is

$$\mathcal{H}_{\text{bulk},[2]}=\mathcal{H}_{\text{op},[2]}\otimes\text{Sym}^*\mathcal{H}_{\text{cl}}=(\mathcal{H}_{1'1}\otimes\mathcal{H}_{2'2}\oplus\mathcal{H}_{1'2}\otimes\mathcal{H}_{2'1})\otimes\text{Sym}^*\mathcal{H}_{\text{cl}}. \tag{6.1}$$

The dynamics leads to transitions between configurations of type $1'1+2'2$ and those of type $1'2+2'1$. Such transitions are suppressed by one factor of e^{-S} (where S is the entropy); see fig. 19.

3) The Boundary Hilbert Space

To specify a state via boundary data now requires a pair of strings labeled by their endpoints, for example $\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}$ or $\mathsf{S}_{1'2}\times\mathsf{T}_{2'1}$. To define the adjoint of a pair, we apply the adjoint operation that was introduced in section 3.3 to each string separately, exchanging its left and right endpoints, and formally replacing a string with the adjoint string. So the adjoint of, for example, $\mathsf{S}_{1'2}\times\mathsf{T}_{2'1}$ is $\mathsf{S}^\dagger_{21'}\times\mathsf{T}^\dagger_{12'}$ (the adjoint strings correspond to bras rather than kets and the right boundary is written first). To each such pair we formally associate a state $\Psi_{\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}}$ or $\Psi_{\mathsf{S}_{1'2}\times\mathsf{T}_{2'1}}$. We refer to such pairs or states as being of type $1'1\times2'2$ or $1'2\times2'1$, as the case may be. Inner products of these states, for example $\langle\Psi_{\mathsf{W}_{1'1}\times\mathsf{X}_{2'2}}|\Psi_{\mathsf{U}_{1'1}\times\mathsf{V}_{2'2}}\rangle$ or $\langle\Psi_{\mathsf{W}_{2'1}\times\mathsf{X}_{1'2}}|\Psi_{\mathsf{U}_{1'1}\times\mathsf{V}_{2'2}}\rangle$, are defined in the obvious way by gluing one string to the adjoint of the other and summing over all possible fillings (fig. 20). With this rule, inner products between states created by string pairs of opposite type are nonzero but are suppressed by one factor of e^{-S}, as illustrated in the figure. That these inner products are positive semi-definite follows from an embedding in the bulk Hilbert space, as discussed shortly. Given this, the boundary Hilbert space $\mathcal{H}_{\text{bdry},[2]}$ for a universe with two open components and any number of closed components is defined in the usual way by dividing out null vectors and then taking a completion to get a Hilbert space.

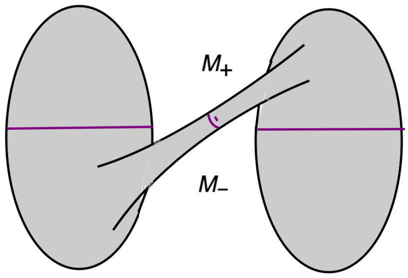

Figure 21. A minimal geodesic cut that separates the region M_- below the cut from the region M_+ above the cut. In this example, the cut has three components, one of which is "in the wormhole."

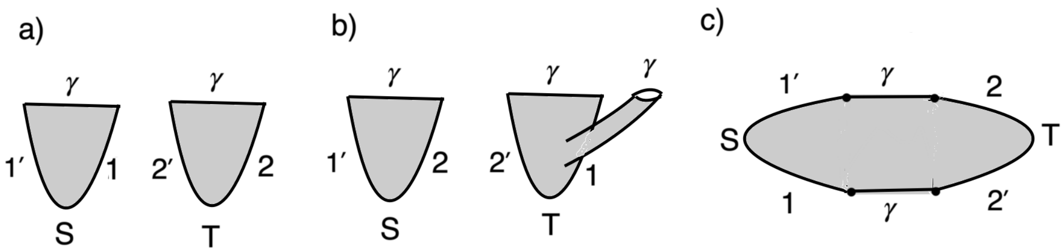

Figure 22. (a) In leading order, the bulk state $\mathcal{W}(\Psi_{\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}})$ created by a string pair of type $1'1+2'2$ is of the same type. In this figure, γ is a minimal geodesic cut with multiple components and the state the state $\mathcal{W}(\Psi_{\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}})$ is a function of fields on γ. (b) In order e^{-S}, $\mathcal{W}(\Psi_{\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}})$ has a component with a closed baby universe. (c) In the same order, it has a component consisting of two open universes of types $1'2+2'1$. As usual, all figures can be decorated with wormholes, and additional closed universes can be added to the final state.

4) Geodesic Cuts By a boundary cut of a spacetime M with several left and right asymptotic boundaries, we mean simply the choice of a point on each asymptotic boundary (if an asymptotic boundary has an endpoint where it meets a geodesic boundary, that endpoint can be part of the geodesic cut). By a geodesic cut γ asymptotic to a given boundary cut of M, we mean a collection of oriented disjoint geodesics that include geodesics that pair up the left and right boundary points in the given boundary cut, together with possible closed geodesics, satisfying the condition that γ divides M into disjoint components M_- and M_+. M_+ is on the side of γ that is specified by the orientation of the right boundaries of M, and M_- is on the opposite side of γ. Pictures are generally drawn to place M_+ "above" the cut and M_- "below" it (fig. 21). If M has geodesic boundaries as well as asymptotic boundaries, then we allow the case that a component of γ is a boundary component of M; in particular, we allow the case that all components of γ are boundary components of M, and M_- or M_+ is empty. A geodesic cut γ is minimal if it has minimal renormalized length among all geodesic cuts asymptotic to a given boundary cut.

5) Boundary to Bulk Map We want to define an isometric map $\mathcal{W}$ from states defined by boundary data (such as a pair of strings) to $\mathcal{H}_{\text{bulk},[2]}$. For example, to define $\mathcal{W}(\Psi_{\mathsf{S}_{1'1}\times\mathsf{T}_{2'2}})$, we sum over spacetimes M that have an asymptotic boundary defined by $\mathsf{S}_{1'1} \times \mathsf{T}_{2'2}$ as well as geodesic boundaries that make up a minimal geodesic cut γ (fig. 22). The dependence of the path integral on the fields on γ then gives a state in $\mathcal{H}_{\mathrm{bulk},[2]}$. Note that by this definition, $\mathcal{W}$ maps a string of type $1'1 + 2'2$ to a bulk state that for large S is mostly of type $1'1+2'2$, but that in order e^{-S} also has a component of type $1'2+2'1$ (fig. 22(c)). As in section 5, the map $\mathcal{W}$ is isometric, that is, it preserves inner products. This is proved by showing that if a geodesic cut γ of M, in, say, fig. 21, is minimal in M_- and in M_+, then it is minimal in M. The proof of this involves the same cut and paste argument as in section 5. Hence, as in fig. 13(c), for string pairs S, T and U, V, if one glues together the path integral construction of $|\mathcal{W}(\Psi_{\mathsf{S}\times\mathsf{T}})\rangle$ and that of $\langle\mathcal{W}(\Psi_{\mathsf{U}\times\mathsf{V}})|$, one gets the same path integral that computes the inner product $\langle\Psi_{\mathsf{U}\times\mathsf{V}}|\Psi_{\mathsf{S}\times\mathsf{T}}\rangle$ between states defined by string pairs, implying that $\langle\Psi_{\mathsf{U}\times\mathsf{V}}|\Psi_{\mathsf{S}\times\mathsf{T}}\rangle = \langle\mathcal{W}(\Psi_{\mathsf{U}\times\mathsf{V}})|\mathcal{W}(\Psi_{\mathsf{S}\times\mathsf{T}})\rangle$. This embedding implies that the inner products of states defined by linear combinations of string pairs are positive semi-definite. Dividing out null vectors and taking a Hilbert space completion, one arrives at the definition of the boundary Hilbert space $\mathcal{H}_{\mathrm{bdry},[2]}$, which then comes with an embedding $\mathcal{W} : \mathcal{H}_{\mathrm{bdry},[2]} \to \mathcal{H}_{\mathrm{bulk},[2]}$.

6) Action of the Boundary Algebra The boundary algebra $\mathcal{A}$ was originally defined as an algebra of operators acting on $\mathcal{H}_{\mathrm{bdry}}$, the Hilbert space accessible to a boundary observer in a universe with just one open component. However, precisely the same algebra acts on both $\mathcal{H}_{\mathrm{bdry},[2]}$ and on $\mathcal{H}_{\mathrm{bulk},[2]}$, and moreover, these actions commute with the map $\mathcal{W}$ between those two spaces. To understand this, consider an observer with access to the left boundary labeled $1'$. To define the action on $\mathcal{H}_{\mathrm{bdry}}$, we start with an obvious action on pairs of strings. We take a string S on the $1'$ boundary to act on a pair of strings in an obvious way, for example $\mathsf{T}_{1'1} \times \mathsf{U}_{2'2} \to (\mathsf{ST})_{1'1} \times \mathsf{U}_{2'2}$. Starting with this action on strings, we want to define an action of S on $\mathcal{H}_{\mathrm{bdry},[2]}$ by (for example) $\mathsf{S}\Psi_{\mathsf{T}_{1'1}\times\mathsf{U}_{2'2}} = \Psi_{(\mathsf{ST})_{1'1}\times\mathsf{U}_{2'2}}$. For this definition to make sense, we need to know that if S is null (meaning that $\Psi_{\mathsf{S}} = 0$ in $\mathcal{H}_{\mathrm{bdry}}$ and therefore $\mathsf{S} = 0$ in $\mathcal{A}$) or $\mathsf{T}_{1'1} \times \mathsf{U}_{2'2}$ is null (meaning that $\Psi_{\mathsf{T}_{1'1}\times\mathsf{U}_{2'2}} = 0$ in $\mathcal{H}_{\mathrm{bdry},[2]}$), then $\Psi_{\mathsf{ST}_{1'1}\times\mathsf{U}_{2'2}} = 0$. The proof of the first statement precisely follows fig. 4 or fig. 15(b), and the proof of the second precisely follows fig. 5(b) or fig. 16(c).

We also want to define an action of $\mathcal{A}$ on $\mathcal{H}_{\mathrm{bulk},[2]}$. This again is done by imitating previous definitions, though the presence of more than one asymptotic component makes the resulting pictures harder to draw or visualize. For example, let us define a matrix element $\langle\Psi'|\mathsf{S}|\Psi\rangle$, where S is a string acting on a specified left boundary and $\Psi, \Psi' \in \mathcal{H}_{\mathrm{bulk},[2]}$. For this, we consider a spacetime M whose boundary consists of an asymptotic boundary labeled by S and two minimal geodesic cuts γ, γ' on which states Ψ, Ψ' are inserted. We assume that γ and γ' each have two noncompact components (along with possible closed components) and therefore four endpoints, and also that γ and γ' have three endpoints in common, and that their fourth endpoints are at opposite ends of S. Some obvious and less obvious choices of M are sketched in fig. 23. The matrix element $\langle\Psi'|\mathsf{S}|\Psi\rangle$ is defined by a sum over all such spacetimes M. The proof that this does give an action of $\mathcal{A}$ on $\mathcal{H}_{\mathrm{bulk},[2]}$ follows fig. 16(b). The presence of an additional open universe component makes the drawing of representative pictures more complicated but does not

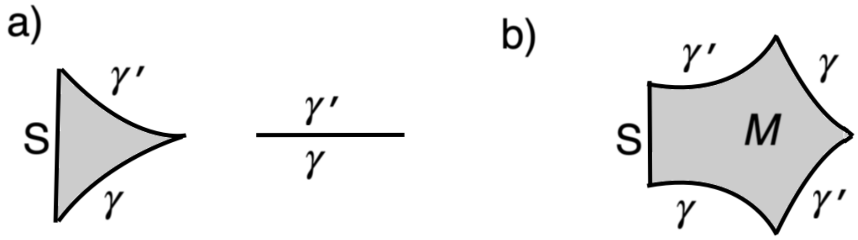

Figure 23. Computation of a matrix element $\langle\Psi'|\mathsf{S}|\Psi\rangle$, where Ψ, Ψ' are bulk states in a world with two open universe components, and the string S acts only on one specified left boundary. (a) The most obvious possibility is that S acts on the state on one open component and does nothing to the state on the other open component. (b) There are less obvious possibilities. In general, the spacetime M has an asymptotic boundary labeled by S and geodesic boundaries labeled by minimal geodesic cuts γ and γ'. γ and γ' have three of their four endpoints in common and the fourth at opposite ends of S. Initial and final states Ψ and Ψ' are functions of boundary data on γ and on γ', respectively. γ and γ' are allowed to have components in common, as in (a).

affect the logic of the argument. Similarly, the proof that the action of $\mathcal{A}$ on $\mathcal{H}_{\text{bulk},[2]}$ and $\mathcal{H}_{\text{bdry},[2]}$ commutes with the map $\mathcal{W}$, in the sense that $\mathcal{W}(\mathsf{S}\Psi_{\mathsf{T}}) = \mathsf{S}\mathcal{W}(\Psi_{\mathsf{T}})$, follows fig. 16(c).

7) Any Bulk State Is Equivalent To A Boundary State Finally, we come to showing that from the perspective of a boundary observer, with access say to a specified left boundary, any pure or mixed state on the bulk Hilbert space $\mathcal{H}_{\text{bulk},[2]}$ is indistinguishable from some pure state in $\mathcal{H}_{\text{bdry}}$. The key picture is fig. 17, generalized now to the case of more than one open universe component (and any number of closed components). By the same logic as before, this picture can be used to show that any pure or mixed state on $\mathcal{H}_{\text{bulk},[2]}$ is equivalent, to a boundary observer, to some density matrix ρ affiliated to $\mathcal{A}$. Setting $\sigma = \rho^{1/2}$, it then follows as before that any pure or mixed state on $\mathcal{H}_{\text{bulk},[2]}$ is actually equivalent for a boundary observer to the pure state $|\sigma\rangle \in \mathcal{H}_{\text{bdry}}$.

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