The Definition of Holography

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Informal Talk

Based on arXiv:2404.00070 [hep-th] and
upcoming work with Aayush Verma

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See also more informal notes on AdS/CFT at https://vkalvakotamath.github.io/teaching/2023-ads-cft-1.

Introduction

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One of the most important revolutions in theoretical high energy physics is that of **holography**, or the duality between a bulk gravitating region and a non-gravitating QFT, usually satisfying conformal invariance.

String theory gives us a way to establish such a duality in the case of anti-de Sitter space, where the bulk is the gravitating AdS and the dual field theory lives on the boundary of AdS. The original description was the duality between type IIB superstring theory and an $\mathcal{N} = 4$ SYM theory, but after that a lot of generality has been found.

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We will try to point out the most general definition of holography and some more technical details in this talk. In the first half, we will discuss what holography generally means and how to approach this phenomenon, and in the second half, we will discuss some more details of a mathematical background to holography.

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We will generally either stick between anti-de Sitter space or de Sitter space holographies.

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Strongly, from bulk reconstruction of bulk fields in the asymptotic boundary limit $r \to \infty$ [from the **extrapolate dictionary** and **HKLL**], we can see that there are only certain bulk fields that can be fully reconstructed given access to a particular boundary subregion. This bulk subregion is called the *entanglement wedge* $\mathcal{E}_W(R)$ for a boundary subregion R.

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Holographic entanglement entropy first indicated this, with Ryu and Takayanagi's famous derivation of the RT formula:

$$S_R = \frac{\text{Area of } \gamma_R}{4G_N\hbar} \,. \tag{1}$$

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Note that usually the bulk is the side we are more ignorant about. But here, we are trying to understand how these subregions imply deeper mathematics.

Axiomatic QFT

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In general, these are referred to as **functorial QFTs**. *Locally covariant QFTs* are functorial QFTs that are functors from the category of globally hyperbolic manifolds to the category of C^*-algebras, satisfying some conditions like *causality*, *isotony*, *covariance*, etc.

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Without going into the details, the idea is basically that there are three types of von Neumann algebras (cf. Gell Mann Murray and von Neumann), loosely based on the tracial properties that are of concern to us.

Bonus Slide: AQFT

To be a little more precise, we consider AQFTs as a functor from the category of $D \geq 2$ globally hyperbolic manifolds Man_D to the category of unital C^* algebras Alg. The objects typically are composed of smooth D-manifolds, with orientation of nowhere vanishing n-forms and time-orientation. The morphisms are smooth isometric embeddings that preserve these orientations.

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In general, all this really means that for every bounded subregion, one can attribute a C^* algebra. More strongly, a von Neumann algebra.

For general subregions with no asymptotic boundary structures, this von Neumann algebra is typically type III_1 with reduction to type II_1 Jensen, Speranza and Sorce.

Types of von Neumann algebras

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Naively there are three types of von Neumann algebras, but there are sub-classifications based on some intrinsic properties of their algebras.

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Bulk QFTs are no exception, and in the Ryu-Takayanagi formula this is rather explicitly clear.

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This is due to a theorem in mathematics by Takesaki:

$$\mathrm{III}_1 \rtimes \mathcal{R}_{\mathrm{aut}} = \mathrm{II}_\infty \, . \tag{2}$$

Crossed Product Algebra

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What happens now?

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Important thing: *we don't have to rely on Euclidean path integrals!* (Path integrals are complicated...)

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Kind of problematic in string theory: vacuas, Maldacena-Nunez, etc. But exciting nonetheless!

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Spoiler alert: It isn't fun.

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Which brings us back to the origianl question: what *is* the most general definition of holography?

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So far the de Sitter picture is not clear in either case, but there are many things to look out for.

Thank you for your attention!