HDA - I

Vaibhav Kalvakota

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In this short talk we will start by talking about TFTs and what n-categories are.

More importantly, *what is the intuition for doing HDA for TFTs?*

TFTs naturally start from *n*-cobordisms and when we attribute to Cobord, $\mathrm{Vect}_{\mathbb{K}}$.

That is, it is a representation of Cobord – a functor from this category to $\mathrm{Vect}_{\mathbb{K}}$ with certain axioms.

Something in common to both: **symmetric monoidal categories**. That is, $\otimes$ "exists" in a way that will be clear in a minute, and the unit objects for both are sent to each other. Further,

$$\mathbf{b}_{y,x}\mathbf{b}_{x,y} = 1_{x\otimes y} \; . \tag{1}$$

Further, they are "rigid" categories.

Modular category $\mathcal{C} \rightarrow$ 3d TFT. (Turaev).

MTC structures match with topological operations – tells you TFT data.

n-TFT loosely requires *n*-categories with **b** and $\otimes$ among other things. (What are these?)

Make it precise *which sorts of n-categories you mean*!

This is where HDA comes in.

The starting point will be to try to define an n-category. We already know a reasonable definition for it, which would be that "it is a category composed of n-morphisms".

A 1-category has morphisms, 2-category has 2-morphisms from morphisms between objects and 1-morphisms, and so on. In such a sense, when we talk about some $A \to B$ for $A, B \in \mathcal{C}$, we would want an equivalence with not only just the 1-morphisms between these objects, but actually $\mathrm{Hom}(A, B)$ category.

When doing TFTs, we want representations of some algebra of interest – hence the "higher dimensional algebras" are of interest.

We have two options here: we can choose from either a *strict* n-category or a *weak* n-category.

(Semistrict categories have been used for 2-categories, from Kopranov-Voevodsky.)

A strict category fundamentally isn't a natural choice, since $A = B$ and $A \cong B$ are fundamentally two different things, since $\cong$ really has to satisfy more and more coherence laws as you have larger n.

In fact, weak n-categories have only been understood for $n < 3$.

Eg. the Yang-Baxter equation in a braided monoidal category:

$$
\begin{array}{ccccc}
 & x \otimes (y \otimes z) & \xrightarrow[\mathbf{b}]{} & (y \otimes z) \otimes x & \\
\swarrow^{\mathfrak{a}} & & & & \searrow^{\mathfrak{a}} \\
(x \otimes y) \otimes z & & & & y \otimes (z \otimes x) \\
\searrow^{\mathbf{b}\otimes \mathrm{id}} & & & & \nearrow^{\mathrm{id}\otimes \mathbf{b}} \\
 & (y \otimes x) \otimes z & \xrightarrow{\mathfrak{a}} & y \otimes (x \otimes z) &
\end{array}
$$

If we went up a level to a bicategory, not only does this figure get extended (since this equation holds to a 2-isomorphism and not just the one figure), you get two different 2-isomorphisms that give you the same figure. (These two 2-isomorphisms turn out to be the same.)

The bottomline is that with weak categories, you really have coherence laws that become harder and harder to make a precise sense of.

But to even make use of "stabilization" (as we shall talk about), we must first identify each n-TFT with a weak n-category. These come with some data, such as the tensor product, braiding operator, etc.

Constructing tensor products: Take C_n with a single object A. The morphisms from A to itself form C_{n-1}, where composition gives a tensor product $\otimes$, making C_{n-1} a monoidal n-category. For braiding, consider C_{n+2} with one object and one 1-morphism. For $n = 0$ (2-category), Eckmann-Hilton gives commutativity $x \otimes y = y \otimes x$. For $n = 1$ (3-category), we get a braided monoidal category with $\mathbf{b} : x \otimes y \to y \otimes x$, but crucially $\mathbf{b}^2 \neq \mathbf{id}$ (non-symmetric), which detects knots. For n=2, you have a braided monoidal 2-category.

The effect of this observation is that primarily, any weak C_{n+k} with single j-morphisms for each k is a monoidal n-category with k-tuples, which give structures like the braiding seen above. The $k = 1$ case has a tensor product but no commutativity, $k = 2$ has commutativity and subsequently $n = 2$ for it would be a braided monoidal 2-category, and so on. This is all too easy to say, since higher than weak 3-categories we lose all senses of coherence laws.

	$n = 0$	$n = 1$	$n = 2$
$k = 0$	sets	categories	2-categories
$k = 1$	monoids	monoidal categories	monoidal 2-categories
$k = 2$	commutative monoids	braided monoidal categories	braided monoidal 2-categories
$k = 3$	‘’	symmetric monoidal categories	weakly involutory monoidal 2-categories
$k = 4$	‘’	‘’	strongly involutory monoidal 2-categories
$k = 5$	‘’	‘’	‘’

Figure: From Baez-Neuchl.