Poincarè Conjecture and the h-cobordism theorem

math.DG Notes
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In this lecture, we will look at the **h-cobordism** theorem and some examples of where it is significant, particularly in understanding the Poincarè conjecture, among many things. Note that I do not cite any specific papers for an introduction, since this lecture points to a number of results for which some technical basic knowledge is presumed.In the next lectures, the references would point to some of the papers used in this one as well.

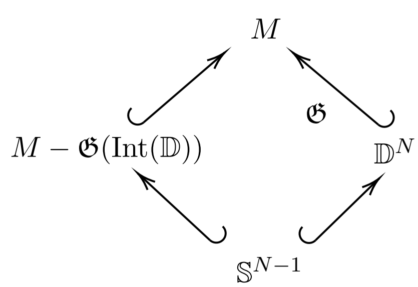
First, we will start by revising what an h-cobordism is. A cobordism is an $N-$Riemann manifold M whose boundary decomposes into embedded smooth manifolds $\partial M = V_0 \sqcup V_1$. An **h-cobordism** (homotopy-cobordism) is one for which the inclusion maps $\iota_0 : V_0 \hookrightarrow M$ and $\iota_1 : V_1 \hookrightarrow M$ are homotopy-equivalences. Then. the h-cobordism theorem can be defined as follows[1]:

Theorem 1 (h-cobordism theorem)**.** Let M be a simply connected $N-$cobordism (with $N \geq 6$) between V_0^{N-1} and V_1^{N-1}. Then, $M \xrightarrow{\cong} V_0^{N-1} \times [0,1]$.

We will apply this in the case of an $N-$disk $\mathbb{D}^N$ for the sake of argument and clarity of application towards the Poincarè conjecture and its proof as it was found by Smale in 1961 through his celebrated $N \geq 6$ h-cobordism technique. In the case of a disk, one can use the same technique, consisting of essentially *splitting* the manifold into primes. We'll start by looking at the following theorem:

Theorem 2. Let M be a contractible manifold satsifying the properties of theorem 1. Then, $M \xrightarrow{\cong} \mathbb{D}^N$.

Proof. Let $\mathfrak{G}$ be an embedding of $\mathbb{D}^N$ into M and identify the interior $\text{Int}(\mathbb{D})$, for which $M - \mathfrak{G}\,(\text{Int}(\mathbb{D}))$ is a cobordism $\partial M \Longleftrightarrow \mathbb{S}^{N-1}$. If we piece these sections back, we would have M from $\mathbb{D}^{N-1}$ and the cobordism $M - \mathfrak{G}\,(\text{Int}\,(\mathbb{D})) \equiv \mathcal{B}$. The following (homotpy) pushout diagram shows this decomposition:



Here, $\mathcal{B}$ is simply connected (due to homotopy cofiber sequence $\mathbb{S}^{N-1} \to \mathcal{B} \to M$), and therefore, one can has

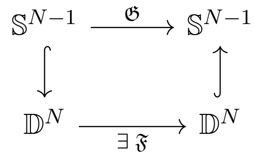
$$\mathbb{S}^{N-1} \xrightarrow{H\cong} \mathcal{B},$$

[1]Note that I defer from providing a proof of this theorem, but one could read this from Milnor's *Lectures on the h-Cobordism Theorem.*

where $A \xrightarrow{H\cong} B$ denotes homotopic equivalence. Now, one can use the h-cobordism result from theorem 1 to identify that $\mathcal{B} \xrightarrow{\cong} \mathbb{S}^{N-1} \times [0,1]$. Gluing $\partial\mathbb{D}$ back, one sees that $M \xrightarrow{\cong} \mathbb{D}^{N-1}$, completing the proof. □

Before stating and working with the Poincarè conjecture, we will first note a rather nice thing:

Remark 1. Given a map $\mathfrak{G}$ between $\mathbb{S}^{N-1}$, one can identify that there exists a homeomorphism $\mathfrak{F}$ between $\mathbb{D}^N$:

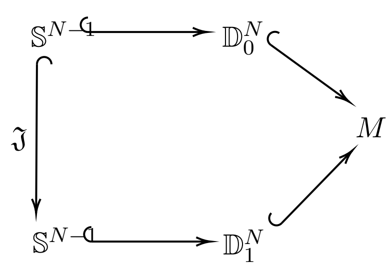


With this clear, we will now state the generalized Poincarè conjecture:

Theorem 3 (A version of the Poincarè conjecture)**.** An $N-$manifold[2] that is homotopy equivalent to the $\mathbb{S}^N$ is also homeomorphic to $\mathbb{S}^N$.

As you can guess at this point, the proof of this is somewhat straightforward in terms of the decomposition of the primes $\mathbb{D}^N_\pm$ and $\mathbb{S}^{N-1}_\pm$ under the assumption of the h-cobordism theorem. The proof is as follows:

Proof. The first diagram shows the decomposition of $M = \mathbb{D}^N_0 \sqcup \mathcal{B} \sqcup \mathbb{D}^N_1$ by identifying the boundaries of the disks and $M - \mathfrak{G}\left(\text{Int}\left(\mathbb{D}^N_0 \sqcup \mathbb{D}^N_1\right)\right) \equiv \mathcal{B}$. Next, using the Alexandrov trick to induce $\mathfrak{F}$ on $\mathbb{D}^N$ from $\mathfrak{G}$ homeomorphism induced on $\mathbb{S}$, we would get the second diagram below: From the second



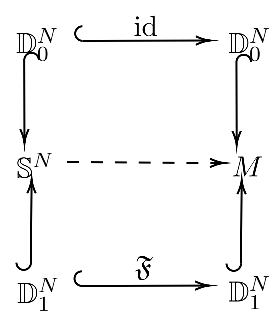


diagram where we used the Alexandrov trick, we see that there is a homeomorphism between $\mathbb{S}^N$ and M by identifying the maps of the $N-$disks. However, in using the h-cobordism theorem, one has to be sure that the inclusion maps *indeed* have the homotopy equivalence nature. This can be found as a lemma as follows:

Lemma 1. If M is homotpic to $\mathbb{S}^N$ and $\mathcal{B}$ is a cobordism between $\mathbb{S}^{N-1}_0$ and $\mathbb{S}^{N-1}_1$ as obtained from the subtraction of the $N-$disk images $M - \mathfrak{G}\left(\text{Int}\left(\mathbb{D}^N_0 \sqcup \mathbb{D}^N_1\right)\right)$, then $\mathbb{S}^{N-1}_0 \hookrightarrow \mathcal{B}$ is a homotopy equivalence.

[2] As we shall see at the end of these notes, by this we mean a *topological* manifold. The smoothness condition poses problems, as one does *not* have a Poincarè conjecture holding from the h-cobordism theorem in the case of smooth manifolds.

Due to the above lemma, one can use the h-cobordism theorem to show that there is a homeomorphism between $\mathbb{S}^N$ and M, concluding our proof. $\square$

Finally, I will remark briefly on the validity regimes of this proof. Let topological manifolds be $\mathbf{Man}^{\mathrm{Top}}$ smooth manifolds be $\mathbf{Man}^{\mathrm{PL}}$, and smooth manifolds be $\mathbf{Man}^{\infty}$. Then, from Rourke and Sanderson, one has the following result:

Theorem 4. In $N \geq 6$, the h-cobordism theorem holds for $\mathbf{Man}^{\mathrm{Top}}$, $\mathbf{Man}^{\mathrm{PL}}$ and $\mathbf{Man}^{\infty}$.

Remark 2. It must be said that our discussion of the h-cobordism theorem is in $\mathbf{Man}^{\infty}$; from the above theorem, it might be instinctive to guess that the Poincarè conjecture also holds for all three. However, this is not true; the Poincarè conjecture does not necessarily hold for $\mathbf{Man}^{\infty}$. To note this, we have the following theorem:

Theorem 5. The Poincarè conjecture holds for $\mathbf{Man}^{\mathrm{Top}}$ and $\mathbf{Man}^{\mathrm{PL}}$ in $N \geq 6$. However, this does not hold for $\mathbf{Man}^{\infty}$.

A very efficient way to note the positivity of the Poincarè conjecture in $\mathbf{Man}^{\mathrm{PL}}$ is to note that the Alexandrov trick is not changed in any way, since the radical extension for a PL map is a PL map. For the $N = 5$ case, we have a *very* special observation – from Donaldson and Kronheimer and Freedman and Quinn, the h-cobordism theorem holds for $\mathbf{Man}^{\mathrm{Top}}$ but does *not* hold for $\mathbf{Man}^{\mathrm{PL}}$ and $\mathbf{Man}^{\infty}$. Reducing one more dimension, in $N = 4$ the Poincarè conjecture and the h-cobordism theorem are the same statement, and as far as I know, there is a problem that is currently open regarding the nature of exotic structures on $4-$spheres. So if any of you have some way to prove it, that would be just amazing.

It must be noted that analytic manifolds $\mathbf{Man}^{\omega}$ *also* enter the discussion. However, I have deferred from adding this into our discussion considering that $\mathbf{Man}^{\omega}$ and $\mathbf{Man}^{\infty}$ are equivalent. This is a theorem due to Grauert and Morrey, which states that *all smooth manifolds admit a unique analytic structure*. Another interesting note over here is that Perelman's proof in $N = 3$ fits in for *all* categories, which follows from the following:

Remark 3. In $N \leq 3$, the manifolds $\mathbf{Man}^{\mathrm{Top}}$, $\mathbf{Man}^{\mathrm{PL}}$ and $\mathbf{Man}^{\infty}$ are equivalent.

Further, a result by Wang and Xu shows that in $N \geq 61$, the smooth Poincarè conjecture we considered above would fail, however this would be a nice thing to provide an outlook of in an upcoming lecture.