Introduction to Poincaré and Ricci flow: Part I

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In this lecture, I will provide a rather informal outlook of the statement of the Poincaré conjecture – informal in the sense that while I could speak from a topological or PDE perspective, I will stick with a very naive outlook of Ricci flow and the Poincaré. The next set of notes have a different structure, and are highlighted in a particularly geometric background. At least in this lecture we will not arrive at the complete structure of the uniformisation theorem in terms of Ricci flow, or discuss Ricci flow in detail for that matter. The purpose of this lecture is simply to motivate the notion of Ricci flow and summarise what we will be doing in the next notes.

It all starts with a Riemannian manifold M – who doesn't love a good Riemannian manifold? And then you say it is smooth and compact, not to mention that it is equipped with a nice metric g for vectors $X, Y \in T_pM$ that defines another nice thing, the Ricci curvature $\mathrm{Ric}^g(p)$ at every point $p \in M$. Naturally one would expect some trivial geometries such as the $2-$sphere or the $2-$Euclidean space to have a constant Ric^g; in other words,

$$\mathbb{S}^2 \implies \mathrm{Ric}^g(p) = +1; \quad \mathbb{R}^2 \implies \mathrm{Ric}^g(p) = 0; \quad \mathbb{H}^2 \implies \mathrm{Ric}^g(p) = -1.$$

This is the statement of the *uniformisation conjecture*[1]: every compact manifold can be set a constant curvature metric, and a corollary of this is that every compact and connected surface is diffeomorphic to a quotient of one of the three constant curvature geometries $\mathcal{C}$. In $N = 1$, the whole question becomes a moo-point, since the only such surface is a circle with no intrinsic geometry to consider. In $N = 2$, the above statement coalesces with the $N = 2$ Poincaré conjecture:

Theorem 1. Every compact and simply connected compact surface is diffeomorphic to the $2-$sphere.

The motivation of the uniformisation conjecture is to arrive at a classification of topological manifolds based on the connected sum decomposition. There are *many* ways to work out the uniformisation theorem, one of them being Ricci flow. In these notes, we will look at a very untechnical outlook of what Ricci flow is first, before arriving at the Poincaré conjecture.

Simply stated, the Ricci flow is the evolution of a Riemannian compact closed manifold under the PDE given some $t \in \mathbb{R}$ for some $g(t)$ with g_0 being the initial metric.

$$\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}^g. \tag{1}$$

By just mere thoughts one can see that there could evolve singularities, where $\mathrm{Ric}^g \to \infty$ in a finite time over the Riemannian manifold, which is where we stop the flow. This is an important thing – we want to work out the uniformisation theorem in terms of this Ricci flow. For example, in $N = 2$ by evolving sufficiently, we want the metric to evolve into a conformal metric described by one of the $\mathcal{C}$ geometries. Consider the homothetic evolution of a sphere into a point-like surface. This evolution is guaranteed by looking at the nature of the metric, and is a part of a class of manifolds called *Einstein manifolds*; for instance, in the solutions to the field equations $G_{\mu\nu} + g_{\mu\nu}\Lambda = 8\pi G T_{\mu\nu}$, one finds that Ric takes the form of $\mathrm{Ric} = \frac{2\Lambda}{N-2} g_{\mu\nu}$, which is an $N-$dimensional vacuum Einstein metric solution to the field equations $+\Lambda$. Simply stated, if one has

$$\mathrm{Ric} = \lambda g_0 \quad \text{and} \quad \lambda \in \mathbb{R},$$

[1]Throughout these notes we will assume that the surfaces are also smooth, unless where specified.

then it is straightforward to see that the metric takes the form of

$$g(t) = (1 - 2\lambda t)\, g_0.$$

This allows one to see the final evolution of the metric; for instance, for a sphere this would tell us the time after which the evolution becomes point-like. For a hyperbolic space, this would be a positive evolution in that the manifold would homothetically expand.

There is a particularly nice theorem by Hamilton 1988, Chow 1991 and Chen-Lu Tian 2005, which goes as follows:

Theorem 2. If the topology of M is a sphere, there exists a singularity in finite time Ricci flow and the (**renormalized**) manifold becomes that of $\mathcal{C}$ at the blow-up time.

This implies the uniformisation theorem, although the original Hamilton-Chow argument was based on the uniformisation theorem. In Chen-Lu Tien's work this was removed and the uniformisation theorem was implied. But first, let us talk about what the "renormalized" in the above theorem means.

A **Ricci soliton** is some pair (M, g) such that for some vector field $X(t)$, one has the following:

$$\text{Ric} = \lambda g - \frac{1}{2}\mathcal{L}_X g, \tag{2}$$

where $\mathcal{L}$ is the Lie derivative. Define $\sigma(t) = 1 - 2\lambda t$ and let $X(t) = \frac{1}{\sigma(t)}Y$, we get a set of diffeomorphisms ψ_t^* with $\psi_0^* = \text{id}$ and the star being the pullback. Then, define

$$\tilde{g}(t) = \sigma(t)\psi^*(t)\left(g(t)\right), \tag{3}$$

which is a Ricci flow:

$$\frac{\partial}{\partial t}\tilde{g}(t) = -2\text{Ric}^{\tilde{g}}. \tag{4}$$

The sign of λ tells us the nature of the Ricci soliton, i.e. *steady*, *expanding* or *shrinking* based on $\text{sign}(\lambda) = 0$, $+1$ or -1 respectively. A Ricci soliton whose Y can be expressed as $\text{grad}\,(f)$ for $f : M \to \mathbb{R}$ is a **gradient** Ricci soliton, and for this the Hessian satisfies the following:

$$\text{Hess}_{g_0}(f) = \lambda g_0 - \text{Ric}^{g_0}, \tag{5}$$

where we used the fact that $\mathcal{L}_Y g_0 = 2\text{Hess}_{g_0}(f)$. Hamilton's cigar soliton is an example of a steady gradient soliton that opens up cylindrically at infinity – see the reference [CigarSoliton] in **/mathDG/Poincare** to see a graphic demonstration of the. In $\mathbb{R}^3$, the Bryant soliton is an example of a steady gradient soliton that opens up paraboloid-ically at infinity.

Consider a case where a singularity occurs *locally* such that the region can be identified and removed so that the Ricci flow can be continued. If there is a set $\mathcal{O}(t) \subseteq M^N$ such that $\mathcal{O}(t) \cong \mathbb{Q}\left(\mathbb{S}^{N-1} \times \mathbb{I}\right)$ (where $\mathbb{I}$ is an closed interval in $\mathbb{R}$), then M^N is said to have a *neckpinch* if the pullback approaches the shrinking cylinder soliton. We can then identify *surgery*, where we remove the particular patch so that the flow can be continued. This is essentially prime decomposition, and finally we arrive at two prime manifolds. One could now argue that this decomposition might take forever, i.e. the prime manifolds that are found as the connected sum of this surgery might take forever to actually arrive at. However, Perelman argued that this is not so – in a finite time, one should be able to find only a finite number of these decompositions.

But why are we bothered about this? The fundamental reason one is even concerned with this in the first place is due to Thurston's **Geometrization conjecture**, which states that that every smooth oriented compact $3-$manifold can be broken down to a finite decomposition of the eight Thurston geometries of these sums. That is, every $3-$manifold can be constructed by a finite *gluing* of the eight Thurston geometries:

$$\mathbb{S}^3, \quad \mathbb{R}^3, \quad \mathbb{H}^3, \quad \mathbb{S}^2 \times \mathbb{R}, \quad \mathbb{H}^2 \times \mathbb{R}, \quad \mathcal{U}_C\left(SL_2 R\right), \quad \mathcal{H}(\mathbb{R}^2), \quad \mathrm{Solv}(\mathbb{S}^2).$$

Thurston had verified this conjecture for a class of manifolds called *Haken manifolds*, which we would discuss in the next notes.

An observation I think we can include here is that one could say that there are manifolds with constant negative curvature such that the Ricci flow does not lead to a singularity in finite time in the first place; in this case, one can use the Sachs-Uhlenbeck formulation of non-trivial $\pi_2(M)$ to show that minimal spheres exist. In the case of Colding-Minicozzi and Perelman, as long as $\pi_3(M)$ is non-trivial, such singularities would form in finite-time, and from Hurewicz's theorem, one can be sure that both $\pi_2(M)$ and $\pi_3(M)$ cannot be trivial, due to which one has the necessary existence of finite-time singularities. Then, one can show that under Ricci flow the area of these minimal spheres goes to zero, resulting in a finite-time singularity for all simply-connected manifolds. The next stop would be that of showing that the surgery can be done finitely, and due to this the Ricci flow will vanish in a finite-time evolution of the manifold. This has some subtleties, such as the problem the injectivity radius must *not* go to zero faster than allowed. This was solved by Perelman's non-collapsing theorem, which has a lower bound on the injectivity radius in terms of a bounded Ricci curvature. There are several other interesting points, which we will discuss in part-II continuation of this lecture.

The present situation seems somewhat messy, but ultimately, this can be summarised as follows: *given a compact closed* $3-$*manifold, it is diffeomorphic or homeomorphic to the* $3-$*sphere.* At this point, it seems too premature to find a reasonable approach to this, since the previous discussions were seemingly purely naive. However, in the direction in which we would be going in (as well as the direction in which any attempt at understanding Ricci flow and Poincaré would lie in) can be outlined as follows[2]:

(1) The maximum principle,
(2) Summary of existence of parabolic PDE and the nature of Ricci flow,
(3) Existence of Ricci flow,
(4) DeTurck and short-time existence,
(5) On the Uhlenbeck Trick,
(6) Ricci flow as a gradient flow
(7) Compactness, convergence and other aspects of Riemannian manifolds,
(8) Curvature aspects of Ricci flow,
(9) On $3-$manifolds with non-negative Ricci curvature,
(10) $\mathcal{W}$ entropy functional,
(11) The $\mathcal{F}$-functional.

[2]This is not necessarily in a chronological order, however subsequent updates to the order may take place.