Penrose diagrams
General relativity and cosmology, Fall 2022, Lecture 5

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When discussing general relativity, it is helpful to look at the nature of the causal structure – or at least in particular, the distinguishing features of causal and acausal curves. The picture is simple – it would be in terms of a lightcone $\mathcal{L}(p)$ at the point p, and events in the future (respectively past) lightcone would be depicted by $\mathcal{L}^{+}(p)$ (respectively $\mathcal{L}^{-}(p)$). Then, the usual causal definitions follow: a future-directed (respectively past-directed) causal curve $\gamma^{\pm}$ would be one that relates the point $p \in \mathscr{M}$ to a point $q \in \mathcal{L}^{\pm}(p)$, where $q \in \mathscr{M}$. Naturally, an acausal curve would be one that joins p to another point $r \in \mathscr{M}$ which lies outside the $\mathcal{L}(p)$.

In the case of actually looking at the causal structure of solutions to the Einstein field equations, it is impossible to capture the entire causal structure of the spacetime into a finitely defined diagram. However, this is possible by first mapping the causal structure of the lightcone into the regions it defines – that is, we can first define the regions that the lightcone defines in terms of causal curves. From figure 1, it is clear that there are three primary set of curves – timelike curves (inside $\mathcal{L}(p)$), null curves (on the surface of $\mathcal{L}(p)$) and spacelike curves (outside the lightcone $\mathcal{L}(p)$). There are then five categories of curves based on the curves being future-directed or past-directed [1]. Using this division, we can find the infinities in the causal structure, which would be the division defining the causal curves moving to infinity in the future or past directions and a spatial infinity. In order to illustrate this in terms of the causal structure, we will consider the Minkowskian spacetime $(\mathscr{M}, \eta)$, where the metric is

$$ds^2 = \eta_{\mu\nu} dx^{\mu} \otimes dx^{\nu} = -dt^2 + dr^2 + r^2 d\Omega^2 \tag{1}$$

Where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We will extend this into the advanced and retarded null coordinates defined by

$$v = t + r\,, \qquad u = t - r \tag{2}$$

Using this, we can redefine the metric in (1) as

$$ds^2 = -du dv + \frac{(u-v)^2}{4} d\Omega^2 \tag{3}$$

It is a trivial exercise to see that this is not a "disturbed" metric in any way with regard to the causal structure of $(\mathscr{M}, \eta)$. In fact, this can simply be drawn to see that the null curves are still the same, and lie at a $\pi/4$ angle in the $t - r$ plane as illustrated in figure 2.

However, this is still not enough – we still have to resolve the infinities $I^{\pm}$, $i^{\pm}$ and i^{0} into a paper. In order to do so, the easiest way is to consider another coordinate transformation such that it is a Weyl transformation. A Weyl transformation is defined on the basis of a coordinate transformation from ds

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[1] Future or past directed for causal curves $\gamma : I \to \mathscr{M}$ such that the tangent vector $T(\gamma)$ is either timelike or null, and no temporal "directions" for acausal or spacelike curves.

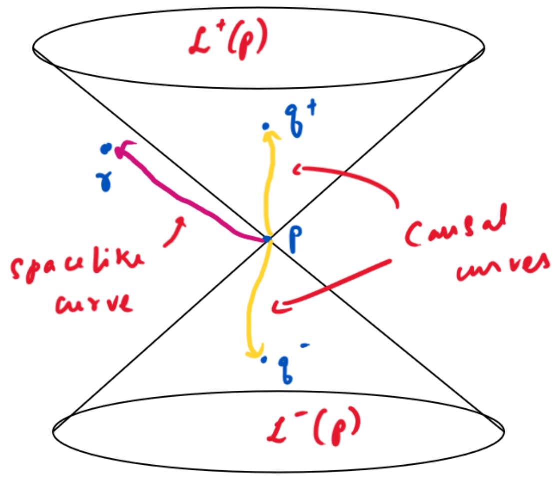

Figure 1: The lightcone structure at $p \in \mathscr{M}$. Forgive me for my terrible drawing.

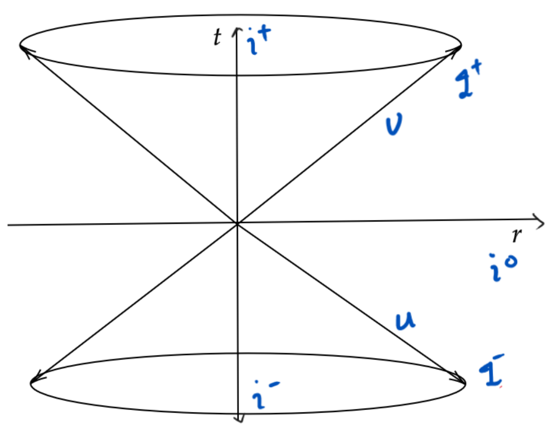

Figure 2: The causal structure of the extended Minkowski metric $(\mathscr{M}, \eta)$ (3)

to $d\bar{s}$ such that both are related via a factor Ω^2 which allows the signs of both the metrics to be the same. In our case, we can choose to use the tangents in terms of u and v:

$$V = \arctan v\,, \qquad U = \arctan u \tag{4}$$

Where u and v lie in the usual tan range $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, we can define the metric $\bar{\eta}$ as the metric defined in these coordinates. The only bit remaining to work out is that of resolving the infinities, which we can do in terms of the tangents and the values relating to u and v. To do this, notice how $u = -\infty$ is simply $U = -\frac{\pi}{2}$ and $v = \infty$ is simply $V = \frac{\pi}{2}$, the usual ranges. Therefore, the infinities can now be represented in terms of an *edge* rather than infinities, which allow us to map the solution onto a paper while preserving the causal structure – this was achieved thanks to the fact that we used a *conformal transformation*, and therefore the null curves were fixed throughout the set of transformations. Therefore, we will be left with a diagram that maps the *entire* causal structure of spacetime into a finitely sized diagram shown in figure 3.

The metric in the new coordinates is given by

$$ds^2 = \frac{1}{\sqrt{\cos U \cos V}} \left(dU dV + \frac{1}{4}\sin^2\left(V - U\right) \right) d\Omega^2 \tag{5}$$

In order to see that this is actually conformally related to the initial form of the metric (i.e. the metric for $\mathscr{M}$ without any coordinate extensions), we can define the coordinates T, R as

$$T = V + U\,, \qquad R = V - U$$

Using these coordinates, the metric in the new coordinates can be written purely in terms of $(T, R, \theta\phi)$ by first noticing that the cosine term $\cos U \cos V$ reduces to $\frac{1}{2}\left(\cos T + \cos R\right)$. Using this, we have the final form of the metric

$$ds^2 = \Omega^{-2}\left(-dT^2 + dR^2 + r^2 d\Omega^2\right) \tag{6}$$

Where Ω is given by the factor $\cos T + \cos R$ – we have written it this way to capture that there is only a factor of Ω^{-2} relating the Minkowskian spacetime $(\mathscr{M}, \eta)$ to the "extended" spacetime given by (6). It is trivial to see that this preserves all of our causal structure from the initial form of the spacetime. Each point in the above diagram is compactified into the surface of the structure, and therefore each point is a two sphere when you draw the surfaces for $t =$ *const.* and so on. Further, the topology of the new metric shows that the Minkowskian spacetime is identical (or conformally related) to the topology $\mathbb{R} \times S^3$.

In the case of black holes, this poses an interesting question – what about the trapped surface and singularity arising in the geometry of a Schwarzschild black hole? The story here is slightly different, and starts with the introduction of the tortoise coordinate r^* (called so after Zeno's paradox of the famous Greek mythological character Achilles and the tortoise), which is defined as

$$r^* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right| \tag{7}$$

This satisfies the slope condition $dr^*/dr = \left(1 - \frac{2GM}{c^2 r}\right)^{-1}$. We now start by defining the null coordinate for the Schwarzschild solution in terms of the coordinates t and r^*:

$$u = t - r^*$$

The Schwarzschild metric is given by the line element

$$ds^2 = -dt^2\left(1 - \frac{2GM}{c^2}\right) + dr^2\left(1 - \frac{2GM}{c^2}\right)^{-1} + r^2 d\Omega^2 \tag{8}$$

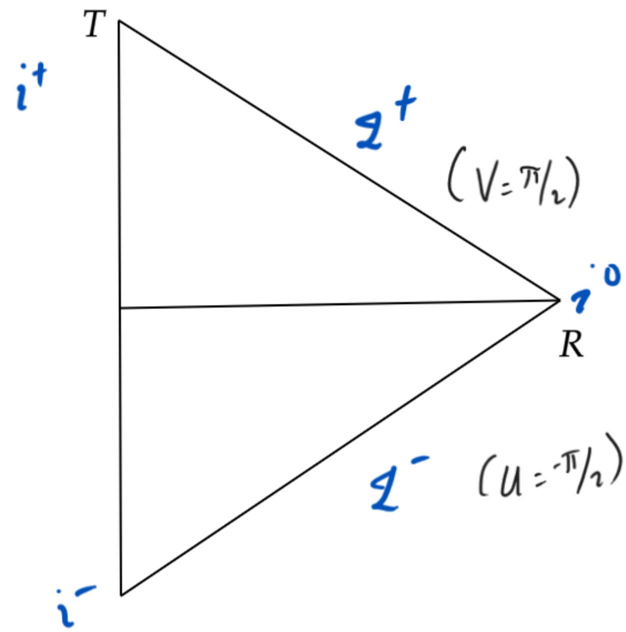

Figure 3: The Penrose diagram for $(\mathscr{M}, \eta)$.

In terms of the new coordinate u, we have

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) du^2 - 2drdu + r^2 d\Omega^2 \tag{9}$$

However, this is not sufficient enough to cover the entire coordinate system as we would naturally expect from a completely compactified diagram of the spacetime. In order to do this, we define the Kruskal-Szekeres coordinates in terms of the interior and the exterior of the black hole (i.e. inside the Schwarzschild radius and outside respectively), which would be of the form

$$T = \sqrt{\left(\frac{r}{2GM} - 1\right)} e^{r/4GM} \sinh \frac{t}{4GM}\,, \qquad R = \sqrt{\left(\frac{r}{2GM} - 1\right)} e^{r/4GM} \cosh \frac{t}{4GM} \quad \textbf{Exterior}$$

$$T = \sqrt{\left(1 - \frac{r}{2GM}\right)} e^{r/4GM} \cosh \frac{t}{4GM}\,, \qquad R = \sqrt{\left(1 - \frac{r}{2GM}\right)} e^{r/4GM} \sinh \frac{t}{4GM} \quad \textbf{Interior}$$

Using these coordinates, we have the *maximally extended Schwarzschild metric*,

$$ds^2 = \frac{32(GM)^3}{r} e^{-\frac{r}{2GM}} \left(-dT^2 + dR^2\right) + r^2 d\Omega^2 \tag{10}$$

Using this, we have the Penrose diagram for a Schwarzschild black hole as shown in figure 4. Note that this black hole was formed by a gravitational collapse, depicted by the curve in the Penrose diagram. The trapped surface is shaded in magenta, and the collapsing interior of the star is shaded in yellow. The red wavy line on the top indicates the singularity formed after the event horizon forms.

The approach we have seen previously can also be used for many other interesting examples of solutions to the field equations, such as the de Sitter spacetime, which is a very interesting and famous solution particularly due to it's use in cosmology. The de Sitter spacetime metric is defined as a solution to the field equations + a positive cosmological constant Λ, which we consider to be zero in most of the elementary cases such as the Minkowskian and Schwarzschild solutions. For defining the de Sitter spacetime, the construction is quite straightforward – you consider the five dimensional Minkowskian spacetime, and consider a four dimensional sub-manifold defined by a hyperboloid:

$$\eta_5 = l^2 \tag{11}$$

Where $l = \sqrt{\frac{3}{\Lambda}}$. Using this, the de Sitter spacetime causal structure can be found by a simple understanding of what the geometry means. The line element

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2 \tag{12}$$

Next, turn all of the X^μ's into the following:

$$X^0 = \sinh\tau\,, \qquad X^1 = \cos\theta\cosh\tau\ldots X^4 = \sin\theta\sin\phi\sin\psi\cosh\tau$$

Using these, we define the induced metric on the four dimensional de Sitter spacetime:

$$ds^2 = -d\tau^2 + \left(\cosh^2\tau\right)d\Omega_3^2 \tag{13}$$

This has a very specific geometric meaning – the spacetime is an infinitely big three-sphere at $\tau = -\infty$, which shrinks to some state at $\tau = 0$ and again reaches to an infinitely huge three-sphere configuration at $\tau = +\infty$. In other words, a D dimensional de Sitter spacetime is an infinitely huge $D-1$ sphere at $\tau = \pm\infty$ and looks like a hyperboloid with a minimum size at $\tau = 0$ as shown in figure 5.

Next, we define a new coordinate $\cosh\tau = \frac{1}{\cosh T}$, which gives us our new metric conformally related to the previously seen metric in D dimensions as:

$$ds^2 = \frac{1}{\cosh^2 T}\left(-dT^2 + d\Omega_{D-1}^2\right) \tag{14}$$

From our previous two examples, we see that the metric preserves causal structure since it is conformally related, and has end points with $T \in \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$. Then, we can draw the Penrose diagram for the de Sitter spacetime as shown in figure 6. As an example to try on your own, try drawing the Penrose diagram for an anti-de Sitter spacetime and a Kerr black hole.

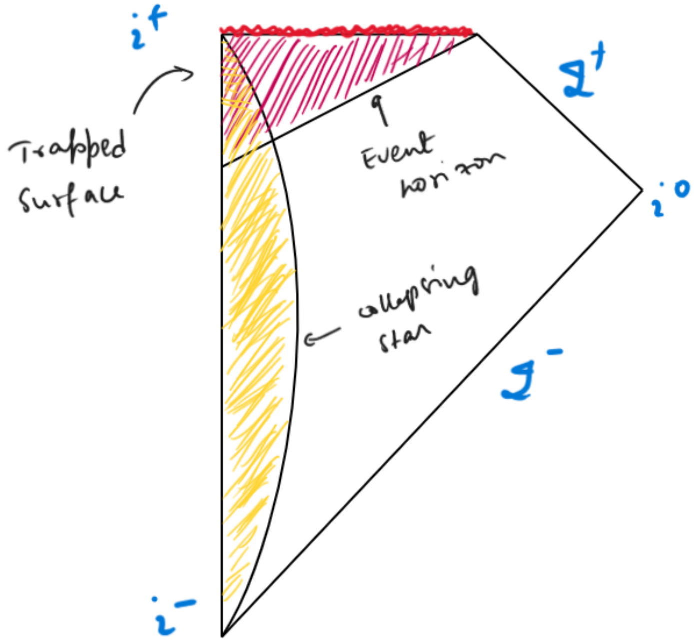

Figure 4: The Penrose diagram for a stellar black hole described by the Schwarzschild solution.

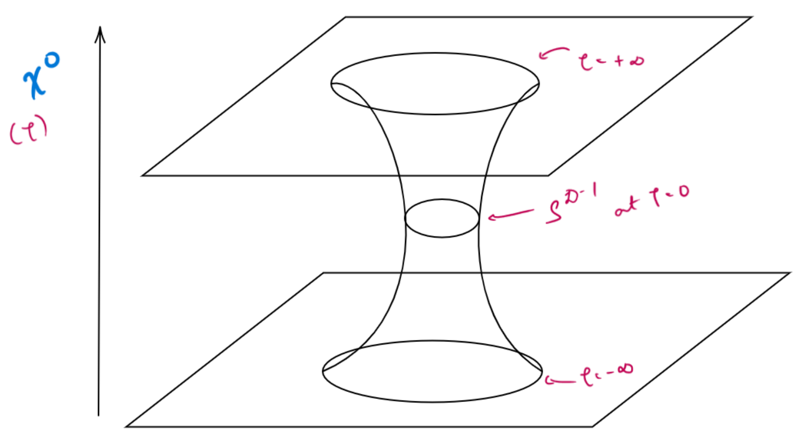

Figure 5: The de Sitter spacetime defined by as a hyperboloid – the minimum size is at $\tau = 0$, which here is the coordinate X^0.

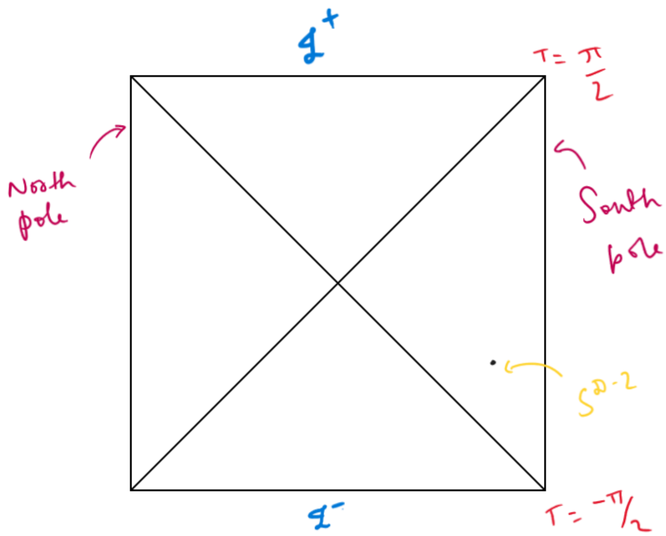

Figure 6: The Penrose diagram for the de Sitter spacetime. The diagonals represent a very important feature, the horizons for an observer at either pole, which tells us what information cannot be observed for a given observer for a choice of position – for instance, for an observer at the south pole, no information past the diagonal stretching from the right top to left bottom (in other words, the diagonal from I^- at the north pole to I^+ at the south pole) can reach the observer at the south pole.