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Bulk Physics, Algebras and All That

Part One: Bulk Reconstruction and Subregions

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Abstract: This is a discussion on some modern aspects of AdS/CFT, with particular focus on entanglement entropy, bulk reconstruction, and operator algebraic aspects of bulk and boundary subregions. Keep in mind that these are a very informal collection of things that I found interesting, and are in no way meant to be a review or formal introduction whatsoever.

In Part One, we will discuss bulk reconstruction and some results surrounding subregions as a motivation towards making sense of tools we will make use of in Part Two and Three. We also mention some details of QES that will be elaborated on more explicitly in Part Two. In Part Two, we will discuss the involvement of quantum extremal surfaces and black hole physics arising from an understanding of subregions and QES. We will talk about QES, purification, pre- and post-Page time behaviour using QES and other related things. In Part Three, we will talk about Jackiw-Teitelboim gravity and Sachdev-Ye-Kitaev model.

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1 Introduction

This discussion paper has grown out of conversations with colleagues on three important aspects of AdS/CFT: (1) holographic entanglement entropy, (2) bulk reconstruction and problems with making sense of bulk-boundary state-operator correspondence, and (3) subregions and associated algebra dualities, which find very important implications in making sense of the emergence of the bulk in AdS holography. There are also some other aspects of things like black holes (in particular the information paradox), Jackiw-Teitelboim (JT) gravity and its relation to random matrix theory (RMT), Sachdev-Ye-Kitaev (SYK) model in exactly solvable models of AdS/CFT in condensed matter, and so on in ese Parts, but none of these will be explicitly pedagogical. Since I do not wish to turn this into a *review* paper, there will not be sections or any specific distinction between sections. Rather, the organisation of this paper will be into two sections – section 2, where we will discuss the three core concepts mentioned above, and section 3, where we will mention two interesting observations on QES that will be useful for motivating Part Two.

But first, for the sake of calling this section an "Introduction", let me explain *why* this discussion is needed, and why these subtleties (or at least some of these things which are subtle) are worthy of yet another discussion in the already vast literature[1]. Most of the times, while discussing aspects of subregion duality or entanglement wedges (as most of the recent things I have been interested in have to do with these), the fundamental motivation is to make sense of the bulk-boundary duality by means of things like bulk reconstruction, operator dictionaries, algebraic aspects involving modular flows and Tomita-Takesaki, and so on. So, for people just getting into this field, an obvious question seems to be, "*so... Is this all about two-point functions on the boundary and in the bulk?*" And while the answer is a yes, there are, like I said, many subtleties that turn up. For example, arguably the most important work in AdS/CFT is that of Ryu-Takayanagi, but the argument clearly paves way to subregions. This is "apparently" obvious, since instead of just the causal wedge $\mathcal{C}_W(A)$ for some boundary subregion A, we now have a much larger patch, the entanglement wedge $\mathcal{E}_W(A)$ to deal with. Now, this will make you go "*oh ok. So you have a huge bulk wedge in which you have a kind of duality... But what duality?*" Which is the point of what people have been working on all this while! To go from Ryu-Takayanagi (RT) [2] to Faulkner-Lewkowycz-Maldacena (FLM) [3] corrections, to Engelhardt and Wall's quantum extremal surfaces (QES) [4] to Jafferis-Lewkowycz-Maldacena-Suh (JLMS) [5] relative entropy result, there are many issues – of which I mentioned a few, like bulk reconstruction, operator dictionaries, modular theory, etc.

Now turn to something that is intermediate – like Hubeny-Rangamani-Takayanagi (HRT) formula. Now, this is nice, but if you think about it a little closely, you will realise that there exists a coarse-graining prescription – found by Engelhardt and Wall – that has some nice features, like relating the holographic "outer entropy" of certain kinds of marginally outer trapped surfaces (called minimar surfaces) to the HRT formula, which originally concerns only extremal surfaces. Naturally one expects extremal surfaces to be a special case of marginally trapped surfaces, but even holographically, we now understand how to make sense of these surfaces and their von Neumann entropies. Which is good. But we still turn out at the same end of the numerous tunnels we could have gone through – "*what to make of bulk and boundary dualities for dealing with explicit operator construction?*" In fact, if you look closely, the RT formula is defined for a very strange case, where we can only make sense of the situation where the lattice spacing $\epsilon \to 0$. Otherwise, we end up in a divergent entanglement entropy situation, which is bad. (This is explained in terms of the type III von Neumann factor nature of the boundary, a point which is made below.)

OK. Now try looking at an apparently whole different problem – what about operator algebras? Maybe we can ask what the von Neumann algebra of the boundary CFT is. Maybe that tells us something about emergence of the bulk – again the

[1] On Inspire (https://inspirehep.net/literature/451647), Maldacena's AdS/CFT paper [1] has 18,908 citations.

same end. Modular flows – Connes cocycle flow – better understanding of Tomita-Takesaki in holography – everything finally ends up at the two core problems: (1) **what are we to make of the explicit bulk-boundary duality?** and (2) **how does the emergence of bulk in AdS/CFT look mathematically?** These two questions are the core notions we wish to discuss in this discussion paper.

Of course, there are certainly other questions in AdS/CFT, that are not necessarily about this whole bulk reconstruction or subregion duality debate. For instance, in JT gravity, one could make sense of the Page time for an evaporating AdS black hole. Or, you could ask about islands and entanglement entropy for an evaporating black hole and the exact post-Page time behaviour, and so on. Or for the condensed matter folks out there, you could be interested in the SYK model – the list goes on. While these are very important (worthy enough of their own review, and I have briefly talked about these in this paper as well), my present understanding is that the question of holographic dictionaries lies far in the fundamentals of AdS/CFT (and holography in general, so this also holds for dS/CFT and flat space holography), and is the star of this paper.

Keep in mind that, since this is not a review of any kind, I have not captured all recent developments. In general, there are many developments that have taken place recently that are not as mainstream as previous works but ones that have a lot of significant implications in AdS/CFT. In particular, one of those results is that of *Cauchy slice holography*, introduced recently by Rifath Khan, Goncalo Regado and Aron Wall [6]. This has many important prospects in AdS/CFT[2].

2 Bulk Reconstruction, Subregions, and Entanglement

The starting point for our discussion is the RT formula, and how this "derives" the notion of bulk reconstruction. We will approach bulk reconstruction twice – first, by the natural expectation of bulk reconstruction from the extrapolate dictionary (which is independent of RT), and second, from the RT formula and FLM corrections to RT. We will also discuss HRT in the same spirit as RT, however it must be in mind that the usual geometric procedure for each prescription are different[3].

[2]Such also exist in de Sitter holography. In fact, more so, since de Sitter holography, as it stands, is very little understood. While there exists dS/CFT due to Strominger [7] and a nice understanding of perturbative quantum gravity, the question of finite bulk physics and deformations is very controversial. So far, there are only two works incorporating Cauchy slice holography with dS/CFT; by Regado [8] and Khan [9].

[3]For RT, we simply calculate the area of a minimal spacelike geodesic γ_A joining the endpoints of the boundary subregion, and FLM corrections arise from an additional bulk corrections. For HRT, we calculate the area of an *extremal* (minimal) surface $\mathcal{X}_A$ homologous to the boundary subregion A in the bulk from the maximin prescription. Semiclassical EW corrections to this arise from bulk corrections, but this is a different calculation from the usual FLM corrections to the RT formula.

2.1 From Ryu-Takayanagi

The RT formula dictates that the entanglement entropy of a boundary subregion A is given by the area of a minimal spacelike geodesic joining the boundary endpoints:

$$S(A) = \frac{c}{3}\log\frac{\gamma_A}{\epsilon} \equiv \frac{\text{Area of }\gamma_A}{4G_N\hbar}\ . \tag{2.1}$$

This can be extended into the HRT prescription, where the area of a bulk (minimal when more than one) extremal surface gives the entanglement entropy. The notion of $\mathcal{E}_W(A)$ in both the cases is still the same – although in the HRT prescription, it becomes more apparent as to what coarse-graining and other problems look like. From the FLM corrections, we can expect what bulk corrections would look like. For HRT, in the sense of quantum extremal surfaces, this becomes the generalized entropy

$$S_{gen} = \frac{\text{Area of }\mathcal{X}_{HRT}}{4G_N} + S_{bulk} + \dots\ . \tag{2.2}$$

This way, one could say that the entanglement entropy of A becomes something like the following:

$$S(A) = S_{gen}(\mathcal{X}_{\mathcal{HRT}})\ . \tag{2.3}$$

This is at all orders in $\hbar$ – that is, sitting in perturbative quantum gravity, we have the following expansions in $g_{\mu\nu}$ for orders in $\hbar$[4]

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1/2)}_{\mu\nu} + g^{(1)}_{\mu\nu} + \dots\ . \tag{2.4}$$

It is not hard to notice that, the $\mathcal{X}_{HRT}$ surface lies deeper in the bulk than the causal wedge $\mathcal{C}_W(A)$. Indeed, recall that the causal wedge is union of the future and past domains of dependence – that is, $\mathcal{C}_W(A) = I^+(D^+(A)) \cup I^-(D^-(A))$. In fact, it can be shown that $\mathcal{X}_{HRT}$ *always* lies deeper in the bulk than $\mathcal{C}_W(A)$. This is a very subtle yet important result – if one picks the entanglement wedge w.r.t this surface, clearly the picture of understanding bulk operators becomes affected. Since $\mathcal{E}_W(A)$ lies deeper in the bulk than $\mathcal{C}_W(A)$, this is a larger bulk wedge in which we can ask the duality question – "*given a bulk operator, what is the dual CFT operator?*"

Let me emphasise that the question of HRT in itself is a much larger set than you can expect. For instance, let us ask if the area of a suitable marginally trapped surface is related to the von Neumann entropy in the bulk-boundary prescription. The following definition will be useful.

Minimar surfaces: *A surface is said to be a minimar surface if it is a marginally trapped surface and is homologous to the boundary subregion. Additionally, this homologous surface is also a weaker kind of a minimal HRT surface when extremal.*

[4] That is, contributions from graviton fluctuation expansions.

At this point we can make sense of where we are working in terms of the coarse-graining. It should be obvious that the conformal completion of the spacetime has to be taken after coarse-graining, so as to find the complete coarse-grained spacetime. This could be done by acting on the expansion and other relevant objects (such as the trace of the extrinsic curvature $\mathcal{K}$) with the $\mathcal{CPT}$ conjugation, which would generate the completion (M', g') to the coarse-grained (M, g) wedge. The main result in EW's paper is that the outer entropy for a minimar surface μ is given by a bound in terms of a corresponding HRT surface whose area coincides with μ, implying that

$$S^{outer}(\mu) = \frac{\text{Area of } \mu}{4G_N\hbar} \ . \tag{2.5}$$

In order to make sense of this, we have to keep in mind that to patch μ to the outer wedge $O_W(\mu)$, we have to consider the junction conditions for the initial data across $O_W(\mu)$. This is a somewhat tedious thing to do, but can be done nonetheless. The final construction leading to the core relation between μ and $\mathcal{X}_{HRT}$ is to make use of *representatives*, which are defined for minimar surfaces as $\bar{\mu}(\Sigma) = \mathcal{N}^{\pm}(\mu) \cup \Sigma$, where $\mathcal{N}^{\pm}$ are null congruences in the orthogonal outward null directions $k^{\mu}_{\pm}$. From the null energy condition (referred to as the null *convergence* condition in the paper), the area of the representative is bounded to μ as Area of $\bar{\mu} \leq$ Area of μ. By definition, outer entropy is the *maximized* entropy corresponding to some ρ attributed to μ, and we get

$$S^{outer}(\mu) \leq \frac{\text{Area of } \mu}{4G_N\hbar} \ . \tag{2.6}$$

Eventually, one can show that a unique extremal surface exists that defines the HRT surface, and by $\mathcal{CPT}$, one obtains the complete spacetime from the auxiliary spacetime (M', g') identified to match the extremal surfaces. This gies us the equality as a saturation of (2.6).

Now, to come back to the topic, keep in mind that the results in the above case is linked to that of reconstruction – of course, in the above discussion there was no need to worry about bulk-boundary operator dictionary. However, the general idea is that when one has a particular $\mathcal{E}_W(A)$, it must also be possible to reconstruct operators in the bulk given an understanding of the dual CFT operators. While the overall discussion did not assume an understanding of bulk reconstruction (or in fact any operator dictionary), the question as to "*can you identify the operators in $\mathcal{E}_W(A)$?*" still remains. In fact, one can motivate this from RT itself, without appealing to HRT! However, given that HRT in general is associated with the bulk more intrinsically than RT, we take it that the HRT prescription also requires us to understand bulk reconstruction and subregion duality in general.

2.2 Extrapolate dictionary

Now we will take a parallel but slightly distinct route to bulk reconstruction. Recall that the general idea of AdS/CFT is that operators in the bulk are dual to operators in the CFT by a conformal weight Δ. The extrapolate dictionary is

$$\lim_{r\to\infty} r^{\Delta}\Phi(r,x) = \mathcal{O}_{\Phi}(x)\ , \tag{2.7}$$

which is to say that taking bulk operator insertions at the boundary gives us some dual CFT operator $\mathcal{O}_{\Phi}$. Then, one can also use the extrapolate dictionary in the sense of n-point functions by (2.7):

$$\lim_{r\to\infty} r^{n\Delta}\langle\Phi(r_1,x_1)\dots\Phi(r_n,x_n)\rangle = \langle\mathcal{O}_{\Phi}(x_1)\dots\mathcal{O}_{\Phi}(x_n)\rangle_{\mathrm{CFT}}\ . \tag{2.8}$$

Keep in mind that the bulk operator insertions are *necessarily* at the near-boundary limit for the extrapolate dictionary. However, for the moment this will suffice. The Hamilton-Kabat-Lyfschytz-Lowe (HKLL) proposal from 2006 states that this dictionary can be used to reconstruct bulk operators by taking a bulk field Φ inserted at some $\mathbf{Y}$, and then taking the double-lightcone of this. Then, the spacelike patch that is bounded by the double-lightcone, say $\mathbf{S}$ contains a *smearing* function $K(\mathbf{Y}|x)$ – this Green's function has a support in $\mathbf{S}$, and $\Phi(\mathbf{Y})$ can be expressed as:

$$\Phi(\mathbf{Y})\int d^Dx\ f_{\Delta}(\mathbf{Y}|x)\mathcal{O}_{\Phi}(x) + \dots \mathrm{O}(1/n)\ . \tag{2.9}$$

The interesting thing is that at leading order in $1/N$, the bulk locality can be expressed in terms of factorization in large N limit. The function $K(\mathbf{Y}|x)$ can be found by inverting (2.9). In this way, one comes back towards the question of reconstruction. Initially, like stated in the previous subsection, the problem seemed to be for $\mathcal{C}_W(A)$, which in itself is only a part of the bulk that can be reconstructed from CFT operators. However, after HRT, the picture was found to be incomplete, being a part of the entanglement wedge $\mathcal{E}_W(A)$. So, to make sense of these things, some notion of reconstruction must be first motivated – so that the notion of subregion-subregion duality can be found, either mathematically or physically. Of course, the latter is much more intuitive than the former, as we shall see soon. As of now, we wil shift our focus from purely talking about bulk reconstruction to a quick understanding of Quantum Error Correction (QEC).

2.3 Reconstructing $\mathcal{E}_W$ and the Code Subspace

The reconstruction of the bulk operators does not span the full CFT Hilbert space $\mathcal{H}_{CFT}$, but instead a subspace of it, referred to as the *code subspace*. In the scheme of entanglement wedge reconstruction, which is a subset of bulk reconstruction, there are primarily two subregions associated to each of the boundary and bulk side – the boundary subregion A and its complement $\bar{A}$, and the corresponding bulk subregion a and its complement $\bar{a}$. To see how this looks like in $\mathcal{E}_W$ reconstruction, identify the bulk-boundary duality in terms of the factorizing Hilbert spaces. That is, first identify the CFT Hilbert space as

$$\mathcal{H}_{CFT} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}\ . \tag{2.10}$$

From this, identify that the corresponding entanglement wedges factorize into Hilbert space

$$\mathcal{H}_{code} = \mathcal{H}_a \otimes \mathcal{H}_{\bar{a}} \ , \tag{2.11}$$

where $\mathcal{H}_{code}$ is the code subspace. The duality between A and a can be used to say that the complements also have a duality, i.e. $\bar{A}$ is dual to $\bar{a}$. This way, the operators in a and the CFT operators in A are dual, whereas bulk operators in $\bar{a}$ are dual to the CFT operators in $\bar{A}$. However, it should be clear that if we consider something like a two-component boundary subregion $A = A_1 \cup A_2$, we would have two extremal surfaces such that *two* bulk subregions would be dual, say a_1 and a_2. Since the corresponding entanglement wedges become $\mathcal{E}_W(A_1)$ and $\mathcal{E}_W(A_2)$, one could ask if by increasing A we could reconstruct an operator that lies deeper in the bulk than the entanglement wedges, since the complements $\bar{A}_1$ and $\bar{A}_2$ instead become smaller. This has some interesting aspects, having to do with *partial* code subspace sharing between A and $\bar{A}$, but we will defer a discussion on this to Appendix A.

This aspect of the entanglement wedge reconstruction and operators in the sense of the reconstruction theorem due to Dong, Harlow and Wall (DHW) will be discusse later in our discussion of the Connes cocycle flow, JLMS and one-sided modular flow. For now, we will discuss the involvement of modular flows and modular Hamiltonians, which will motivate our discussion of the JLMS result for relative entropy equivalence between bulk and boundary sides.

2.4 Modular flows and Hamiltonian

As stated previously, at leading order in $1/N$, bulk locality can be expressed in terms of large N factorization. The entanglement wedge reconstruction aspect has a nice aspect in the discussion, since we can cover it in AdS-Rindler coordinates and describe an HKLL-type reconstruction. We get something like the following, taking a vacuum state in CFT:

$$\Phi(\mathbf{Y}, \mathcal{E}_W(A)) = \int_{\mathcal{D}(A)} dx \ f_{\Delta}^{Rindler}(\mathbf{Y}|x)\mathcal{O}_{\Phi}(x) \ , \tag{2.12}$$

where $\mathcal{D}(A)$ is the double-lightcone system.

We can now define a *modular flow* based description of (2.12) by describing a modular flow on basis of ρ_A (dropping the Φ subscripts on all $\mathcal{O}$'s)

$$\mathcal{O}_s(x_A) = \rho_A^{-is/2\pi}\mathcal{O}(x_A)e^{is/2\pi} \ , \tag{2.13}$$

for which equation (2.12) becomes modified to include $\mathcal{O}_s$ in (2.13). To motivate what will be the case in the JLMS prescription, we will discuss aspects of the modular Hamiltonian as well. Define a modular Hamiltonian K_A as:

$$K_A = \frac{\mathbf{A}(\partial a)}{4G_N} + K_{bulk,\,a} + \mathrm{O}(G_N) \ , \tag{2.14}$$

where $K_\rho = -\log\rho$ (we will revisit this later as well). (2.14) can also be seen to motivate the equivalence between the commutators of K_{bdy} and K_{bulk} with bulk fields, which is essentially the motivation to the JLMS formula, which a little more explicitly takes into consideration of the *relative entropy*.

As mentioned in the earlier discussion on HKLL reconstruction, one can take the extrapolate dictionary (2.7) and the invert construction (2.9) for a bulk-boundary correlator, to find the function $K(\mathbf{Y}|x)$. In this way, this distribution can be formally understood. To see how this all adds up, one can take the above mentioned equivalence of the modular Hamiltonian on both sides with the $1/N$ corrections:

$$[\Phi(\mathbf{Y}_A), K_{A,\,bdy}] = [\Phi(\mathbf{Y}_A), K_{A,\,bulk}] + \mathrm{O}(1/N\ ,) \tag{2.15}$$

which gives the bulk modular flow that Faulkner and Lewkowycz found:

$$e^{iK_A s}\Phi(\mathbf{Y})e^{iK_A s} = e^{iK_{A,\,bulk}s}\Phi(\mathbf{Y}_A)e^{iK_{A,\,bulk}s}\ . \tag{2.16}$$

This has some subtleties with the exponential of the modular flow and modular Hamiltonian, which can be put aside in the present discussion in the sense of low energy theory. One can then explicitly make sense of this modular flow and the exponentiated JLMS formula, although a very formal discussion on this has not been provided in this paper (see section 2.6 for an informal set of remarks on this).

Finally, we will discuss the notion of relative entropy and why it is natural to find relative entropy in AdS/CFT. As we shall discuss, relative entropy in a sense is more fundamental than the usual entanglement entropy computed by the analytic continuation $n \to 1$ of Renyi entropy S^n due to the type III nature of general local quantum field theories. We will discuss relative entropy in Araki's approach in terms of Tomita-Takesaki theory.

2.5 Relative entropy and Tomita-Takesaki

We will first start by mentioning the somewhat straightforward aspects of relative entropy. For some ρ and σ, we have the relative entropy defined as

$$S(\rho|\sigma) = \mathrm{Tr}\rho\log\rho - \mathrm{Tr}\rho\log\sigma\ . \tag{2.17}$$

One could then find the properties of $S(\rho|\sigma)$, such as monotonicity, etc. However, for our purposes, this is not a very *solid* definition. A better approach is to use Tomita-Takesaki theory.

Let $|\Psi\rangle \in \mathcal{H}_{QFT}$ be a an excited state w.r.t some $|\Omega\rangle$. Then, the relative Tomita operator looks like

$$S_{\Psi|\Omega;\mathcal{A}}(\alpha|\Psi\rangle + |\chi\rangle) = \pi(\Psi)\alpha^\dagger|\Omega\rangle\ , \tag{2.18}$$

where $\chi \in (1-\pi'(\Psi))\mathcal{H}$ and in the cyclic separating case (although this is usually not a condition), $\pi(\Psi)$ is the unit operator. The relative modular operator is defined as

$$\Delta_{\Psi|\Omega} = S^\dagger_{\Psi|\Omega}S_{\Psi|\Omega}\ . \tag{2.19}$$

The relative entropy then is a quantity that is definite, i.e.

$$S(\Psi|\Omega; \mathcal{A}) < \infty \ . \tag{2.20}$$

From Tomita-Takesaki, define an antilinear unitary $\mathcal{J}$ and Δ, which is a positive self-adjoint operator. Then, the relative entropy is defined as

$$S(\Phi|\Psi) - \langle\Psi| \log \Delta_{\Phi,\Psi} |\Psi\rangle \ . \tag{2.21}$$

This way, let ϕ and ψ be two faithful and normal positive functionals in a von Neumann algebra $\mathcal{A}$, and let Φ and Ψ be the cyclic separating vector representatives of respectively ϕ and ψ. Then, $\Delta_{\Phi,\Psi}$ is the relative modular operator in (2.21), which on evaluation gives us (2.17).

In general, relative entropy is a naturally defined quantity for type III von Neumann factors, which is the von Neumann factor of local QFTs. We will also see later, on the topic of type III factors, that the emergence of bulk causality in AdS/CFT is due to a type III_1 von Neumann algebra on the boundary side. However, for now, we will discuss the nature and implications of the modular Hamiltonian.

As said previously, the nature of (2.14) can be used to find that there is an equivalence between the commutator with bulk fields:

$$[K_{bdy}, \Phi] = [K_{bulk}, \Phi] \ . \tag{2.22}$$

This in itself is half of the picture of JLMS formula in the context of operator dictionaries between a boundary subregion and its bulk dual. However, to give this the explicit information theoretic description, we have to make sense of relative entropy associated to A and a.

2.6 JLMS and Bulk reconstruction

We are now in a position to talk about the full nature of the JLMS argument [5]. For the derivation of the JLMS result, I will be referring to Dong, Harlow and Wall's paper [10].

Start by noting that the relative entropy $S(\rho|\sigma)$ can be written as

$$-S(\rho) + \text{Tr}(\rho K_\sigma) \ . \tag{2.23}$$

In order to proceed, we will make use of the so-called *first law of entanglement*, where for some $\rho \to \rho + \delta\rho$, we have

$$S(\rho + \delta\rho) - S(\rho) = \text{Tr}(\delta\rho K_\sigma) + \text{O}(\delta\rho^2) \ . \tag{2.24}$$

Faulkner's paper showed that one can define a bulk operator $\mathcal{A}_{loc}$ given by

$$\mathcal{A}_{loc} = \frac{\text{Area of } \mathcal{X}_{HRT}}{4G_N} \tag{2.25}$$

at leading order in G_N. Using this, one finds that

$$S(\rho_A) = S(\rho_a) + \mathrm{Tr}(\rho_a \mathcal{A}_{loc}) \ . \tag{2.26}$$

Here the n-th Renyi entropy is computed by a bulk path integral to find the analytic continuation $n \to 1$ for finding von Neumann entropy. (2.26) in the sense of quantum extremal surfaces also holds at $1/N$ orders, by extremizing over the generalized entropy contribution. Then, we have

$$S(\rho + \delta\rho) - S(\rho) = \mathrm{Tr}\left((\sigma_a + \delta\rho_a)\mathcal{A}_{loc}\right) + S\left(\sigma_a \delta\rho_a\right) \ . \tag{2.27}$$

Now, we can find the terms like $\mathrm{Tr}(\rho_A K_{\sigma_A})$. This term looks like

$$\mathrm{Tr}\left(\rho_A K_{\sigma_A}\right) = \mathrm{Tr}\left(\rho\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ . \tag{2.28}$$

From the first law of entanglement (2.24), for some perturbation of σ_A in $\mathcal{H}_{code}$, we have

$$\mathrm{Tr}(\delta\sigma_A K_{\sigma_A}) = \mathrm{Tr}\left(\delta\sigma_a\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ . \tag{2.29}$$

All this boils down to

$$\mathrm{Tr}(\rho_A K_{\sigma_A}) = \mathrm{Tr}\left(\rho_a\left(\mathcal{A}_{loc} + K_{\sigma_a}\right)\right) \ , \tag{2.30}$$

which using (2.23) finally gives us the required JLMS formula,[5]

$$S(\rho_A|\sigma_A) = S(\rho_a|\sigma_a) \ . \tag{2.31}$$

For the time being, I will discuss some of the aspects of bulk reconstruction in the sense of subregion-subregion duality. I will not put this discussion as a subsection since this is a naive formulation of what subregion duality is, but in the next section we will delve deeper into some of the mathematical aspects of this, specifically, in the direction of Liu and Leutheusser's work on subregion-subalgebra duality and emergence.

2.7 Subregions and Subalgebras

Bulk locality is a very interesting thing. One usually attributes this to the notion of *emergence* of the bulk in AdS/CFT, although the mathematical aspects of these usually has to do with a very nice aspect of operator algebras associated to the boundary CFT. In general, the idea of the type of von Neumann algebra on the bulk and boundary sides is a very fascinating thing; the boundary CFT is of type III_1, but one can simply introduce a lattice spacing ϵ as discussed previously to turn it into a type I algebra, thereby giving us a nice regulated holographic entanglement

[5] I know that I have skipped over a lot of details, but the point of this derivation is to capture some essentials and not have to reproduce the entire discussion for our purposes.

entropy. Taking into account of $1/N$ corrections, the type III_1 nature becomes type II_∞ – this has also been the center of interest in Chandrasekaran, Longo, Pennington and Witten's work on the algebra of observables in static patch de Sitter holography.

However, a much more mathematically fundamental notion has to do with understanding the subregion-subregion duality of the bulk and boundary. At finitely large N, the bulk Hilbert space $\mathcal{H}_{bulk}$ and the CFT Hilbert space $\mathcal{H}_{CFT}$ are equivalent in the sense of identifying bulk and boundary states equivalently:

$$\mathcal{H}_{bulk} = \mathcal{H}_{CFT} \ . \tag{2.32}$$

In fact, this should not be a surprise – the entire program of bulk reconstruction and operator dictionaries comes from this! From RT itself, one can see that bulk subregions have an interesting property: *they emerge from boundary operator subalgebras.* This is the starting point for our discussion on subregion-subregion and subregion-subalgebra duality. The example of RT is in fact quite fundamental, since the idea of entanglement wedge duality to boundary subregion subalgebra is derived from the fact that the entanglement wedge contains operators that contain information about the boundary subregion. This way, bulk properties like locality arise from purely boundary subalgebra properties – that is, the *bulk is emergent from boundary subalgebra.*

For the sake of discussion, consider also the case of an eternal black hole in AdS. The "right" external region in AdS is dual to operators[6] in the algebra of the "right" boundary in the thermofield double (TFD). In the same way, for some bulk subregion inside the right wedge, we have a dual subalgebra of CFT operators in the corresponding boundary subregion. This extends to all sorts of subregions in AdS/CFT, although keep in mind that the implication so far is that we are working in the large N limit. Subregion-subalgebra duality also is in this limit, although we will not exactly emphasise on this aspect in the discussion.

We will discuss two aspects of subregion duality in this subsection; (1) on entanglement wedge reconstruction in the sense of modular flows [11], and (2) on subregion-subalgebra duality [12, 13]. The former is a motivation towards using more algebraic aspects in the sense of reconstruction and bulk subregion emergence, while the latter is an explicit description of Liu and Leutheusser's work on subregion-subalgebra duality.

Modular flows: We will start from the modular Hamiltonian (2.22). More precisely, let me rewrite this to include the fact that (2.22) is the case at leading order in $1/N$:

$$[K_{bdy}, \Phi] = [K_{bulk}, \Phi] + \mathrm{O}(1/N) \ . \tag{2.33}$$

[6]By this in general we will refer to single-trace operators.

Now, by again ignoring the aspect of exponentiation in JLMS, we will rewrite this in the sense of a modular flow, something like

$$e^{-iK_A s}\Phi(\mathbf{Y})e^{iK_A s} \; . \tag{2.34}$$

Now why would we want to do something like this? Because the modular Hamiltonian has a natural interpretation as the generator of automorphism. This plays a very important role in understanding type III algebras from Tomita-Takesaki as pointed out before. What we could do now is to express an operator $X \in \mathcal{A}_A$ in the form of a modular flow, like

$$X_s = e^{iK_A s} X e^{-iK_A s} \in \mathcal{A}_A \; . \tag{2.35}$$

One can naturally see that the commutator (2.33) looks somewhat fishy – it looks like something you would get also from the boundary side. Then, the duality reduces to something of the form

$$e^{-iK_A s}\Phi(\mathbf{Y})e^{iK_A s} = e^{-iK_{A,\,bulk} s}\Phi(\mathbf{Y})e^{iK_{A,\,bulk} s} \; . \tag{2.36}$$

Now this seems nice. Clearly, this also has some nice subregion aspects, and this is most explicit when considering the case of entanglement wedge reconstruction. However, notice something interesting about the entire argument.

Clearly, the boundary CFT algebra is of type III_1. And this entire argument seems to be intrinsically based on this fact; one could now conjecture that there is a type III operator algebra for such subregion-duality. In fact, one could go ahead and see for the case of RT or eternal black hole case mentioned previously in discussion. If this really were the case, could one enforce this statement by explicitly computing the subregion-subalgebra duality?

Subregion-Subalgebra duality: As said above, the notion of a type III subalgebra on the boundary side seems to correspond to bulk subregion emergence. This is what was referred to as the *subregion-subalgebra duality* in a paper by Hong Liu and Samuel Leutheusser. Simply stated, the following discussion is the key result of their work, which we will discuss in this bit of the subsection.

The starting point now is (2.32). If we start from some CFT state $|\Psi\rangle$, we can express a relation between the bulk Fock space and the GNS-constructed Hilbert space boundary HIlbert space,

$$\mathcal{H}_\Psi^{Fock} = \mathcal{H}_\Psi^{GNS} \; . \tag{2.37}$$

Let us denote by χ (same convention as Liu-Leutheusser) the collection of bulk fields + metric. By χ_Ψ we mean χ corresponding to Ψ, and choose the vacuum state to be $|0\rangle_{\chi_\Psi}$ defined by bulk fields + metric perturbations around $|0\rangle_{\chi_\Psi}$. The example cited in Liu-Leutheusser is that of a bulk field, which looks like [13]

$$\Phi(\mathbf{Y}) = \Phi_0(\mathbf{Y}) + \sum_n \left(u_n(\mathbf{Y})a_n + u_n^*(\mathbf{Y})a_n^\dagger\right) \; . \tag{2.38}$$

Here u_n denote mode functions. The Fock space $\mathcal{H}_{\Psi}^{Fock}$ then is found by operating successively on $|0\rangle_{\chi_\Psi}$ with $a_n^\dagger$. The GNS Hilbert space is obtained using the GNS construction from single-trace operators $\mathcal{S}$'s on $|\Psi\rangle$.

Now, let the algebra of single-trace operators acting on $\mathcal{H}_{\Psi}^{GNS}$ be denoted by $\mathcal{A}_{\Psi}^{GNS}$, and the algebra of bulk fields on $\mathcal{H}_{\Psi}^{Fock}$ be denoted by $\tilde{\mathcal{A}}_{\chi_\Psi}^{Fock}$. Then, looking at (2.37), we must have a correspondence of the form

$$\mathcal{A}_{\Psi}^{GNS} = \tilde{\mathcal{A}}_{\chi_\Psi}^{Fock} . \tag{2.39}$$

This motivates subregion-subalgebra duality. If one picks a bulk subregion a and denotes by $\mathcal{Y}_a$ the bulk algebra, one can notice that in the usual limit it must be of type III_1. From this, one can see from the previous discussions that there must exist a boundary subalgebra $\bar{\mathcal{Y}} \in \mathcal{A}_{\Psi}^{GNS}$ of type III_1, so that we have

$$\mathcal{Y}_a = \bar{\mathcal{Y}} . \tag{2.40}$$

2.8 Connectedness and Algebras

(This discussion is based on Netta Engelhardt's talk [14] on her (upcoming) work with Hong Liu, given at Strings 2023. The discussion is made of musings that I found interesting from the talk, and is not based on a paper as of yet.)

The discussion on QES physics has an interesting aspect, that has been the center of interest from many people – that of *canonical purification*. As mentioned earlier in section 2.1, what one can do is to take a $\mathcal{CPT}$ reflection around a "constructed" (in the sense of EW construction this is the coarse-grained) spacetime with initial data specified. That is, for some initial data on the constructed $\mathfrak{H}(\Sigma, h, \dots)$. By $\mathcal{CPT}$ reflecting $\mathfrak{H}$, one obtains a "mirror" of the initial data on the wedge, giving us a complete geometry from $\Sigma \cup \tilde{\Sigma}$, where Tilde-d quantities are the $\mathcal{CPT}$ reflected ones. With this, the entire coarse-grained spacetime can be constructed by taking these completed wedges $O_W(\sigma) \cup \tilde{O}_W(\tilde{\sigma})$ (glued across the HRT surface by identifying the codimension 2 junction conditions) and evolving them. In the sense of canonical purification, one now first finds the gravity dual for the CP (canonically purified) boundary, and then one does the $\mathcal{CPT}$ reflection across $\mathcal{X}_{HRT}$ to obtain the "full" spacetime. For instance, taking the CFT in a mixed state ρ, the CP-completed spacetime gives the appropriate full Schwarzschild-AdS spacetime. This happens around the QES, which plays a very important role: the connectedness of Σ and $\tilde{\Sigma}$ is determined by this nontrivial QES (see fig:1). However, there are some very interesting algebraic aspects to this.

In general, when one says that the bulk emerges because of entanglement between the two boundaries in TFD, this aspect of connectedness arising from a common QES is the determining factor. If the two boundaries were to be disconnected, reconstruction from $\mathcal{CPT}$ reflection around the QES would not "connect" the two

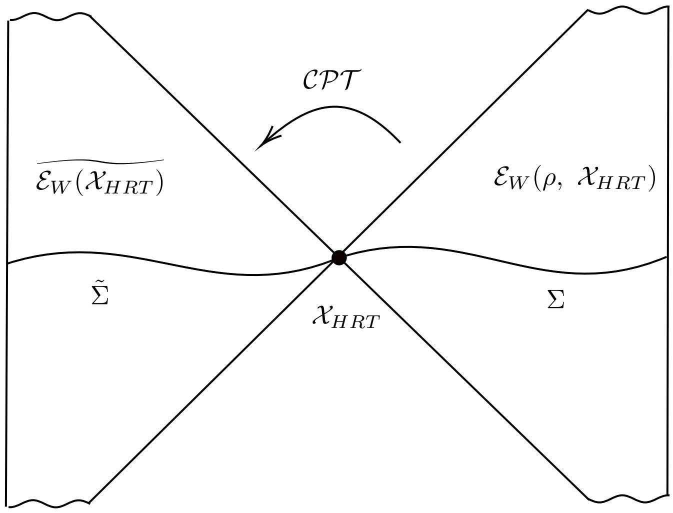

Figure 1. $\mathcal{CPT}$ reflection and continuity around the QES.

boundaries. If two boundary subregions, A_1 on CFT_L and A_2 on CFT_R (where CFT_L and CFT_R depict left and right boundaries respectively), then the entanglement wedges would have a common edge as seen previously. However, as discussed in section 2.7, recollect that in the large N limit, the algebra of bulk operators is type III. The usual way of purifying is to double the Hilbert space, which gives us the TFD state,

$$|TFD\rangle = \frac{1}{Z}\sum e^{-\beta E_n/2}|n_i\rangle|n_j\rangle \ . \tag{2.41}$$

We recover this from the Gibbs state, which gives us $|TFD\rangle$. As said before, the connectedness of this purified state is based on whether or not the full geometry can be constructed from $\mathcal{CPT}$ reflection and evolution from the constraints imposed on $\mathfrak{H}$. If one considers the cases of a pre-Page time and a post-Page time, one can see that the full spacetime geometry in the former case is different from that of the latter. On the basis of connectedness, we see that one is not connected, whereas the other one is connected. In order to better make sense of this, we will use the type III nature and (2.37) that we looked at in section 2.7.

First, start by noticing two interesting details: if the two geometries associated with A_1 and A_2 (and corresponding entanglement wedges $\mathcal{E}_W(A_1)$ and $\mathcal{E}_W(A_2)$) are connected, then they *must* share an edge, as evident from the constructed geometry. Similarly, it must also be that there must not exist states in the bulk Fock space $\mathcal{H}^{Fock}$ that factorize into product states in $\mathcal{E}_W(A_1)$ or $\mathcal{E}_W(A_2)$. From (2.37), it must also be that the same goes for the GNS Hilbert space $\mathcal{H}^{GNS}$. From subregion-subalgebra and emergent type III factors, it is also clear that the operator algebra of the bulk is type III. If the geometry were to be connected, $\mathcal{H}^{GNS}$ must *also* be type III from subregion-subalgebra. Therefore, one can conjecture that the connectedness of these two boundary sides is based on the algebraic emergence in the large N limit. It should also be clear from these observations that if the algebra of bulk operators is type I, then they must be disconnected.

Before remarking further, I wish to briefly take a detour to point out the *no transmission principle* (NTP), which states that if two CFTs are independent, then the corresponding bulk duals must *also* be independent [15]. This has some nice implications, for instance in the strong cosmic censorship situation in AdS/CFT [15, 16], but this also has a bit of relevance to the present discussion on the relation between the type of operator algebras and connectedness.

Start by keeping in mind that the NTP would ordinarily prevent dependent descriptions between two bulks that are dual to two independent CFTs. However, one could similarly argue that there is an algebraic check on this principle. If one wanted to ensure that the two bulk duals were independent, then as seen above, the corresponding bulk algebra of operators must be of type I. This way, one can argue that if the the algebra of bulk operators were type III, then the NTP allows for the connectedness of the two geometries as seen above. Therefore, one could also propose an algebraic description of NTP and algebraic ER=EPR as proposed by Engelhardt and Liu. Of course, the two descriptions are identical, but one could make a more rigorous statement on the NTP by *exactly* showing that this is related to connectedness. However, I will leave this for future considerations.

Finally, the statement of the Engelhardt-Liu speculation of algebraic ER=EPR is that if the algebras $\mathcal{A}(A_1)$ and $\mathcal{A}(A_2)$ associated to A_1 and A_2 are of type III, they are connected. If they are type I, they are disconnected. However, there is an intermediate case here – what about type II algebras? Does there exist a case that satisfies *some* of the properties outlined above, but maybe not all? Clearly, if there were to exist a type II situation, it could be possible that there are divergent entropies but ones for which a trace can be defined.

The pre-Page time case is *exactly* an example we can use here. This cannot be type III or type I, since in the $G_N \to 0$ limit there is clearly a divergence of entropy, but trace can be defined nonetheless. This becomes the type II example we were looking for, and therefore it must also be that this is “somewhat” connected. The speculation by Engelhardt and Liu is that there is a classical ER=EPR description in type III, a quantum connectedness description for type II and no connectedness for type I cases [14]. There is also a further description of phase transitions between bulk subregions being type III, but we will not discuss that here.

I will conclude this discussion by saying that among many important results, I feel that this fundamentally has an interesting statement: that entanglement between two boundaries building the full spacetime is not true has been shown from the above algebraic ER=EPR description, and it would be interesting to see how this looks like in the islands description and the full post-Page time behaviour.

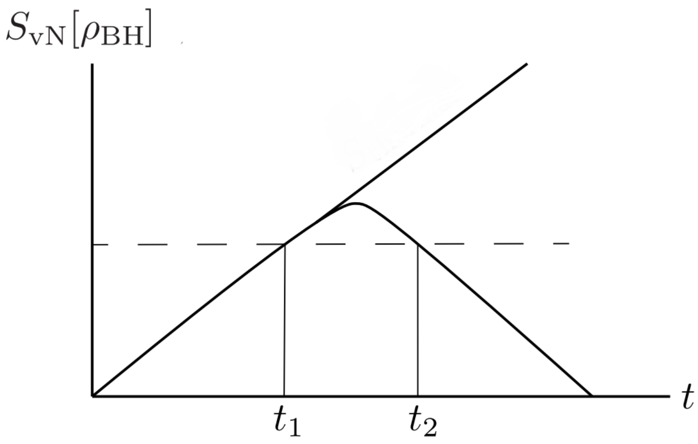

Figure 2. We have a pre-Page time t_1 and a post-Page time t_2 associated to some value of $S_{vN}[\rho_{BH}]$.

3 Quantum Extremal Surfaces

Our discussion in Part Two is based on QES applications to the black hole information problem. We will talk about some of the aspects in which entanglement wedge reconstruction and local bulk operators play a very important role in understanding the information problem in AdS/CFT, and how the nature of QES plays a key role. Keep in mind that the discussions are rather superficial, and I would recommend seeing Suvrat Raju's notes on the information paradox [17] to get a better idea of why AdS/CFT is a good playground for seeing the resolutions to the information problem. See also Aayush Verma's notes on the black hole information problem [18] for an introductory discussion and motivation in a non-AdS/CFT background. As of now, I will mention two points that I feel motivate this discussion significantly.

3.1 QES, Purification and JT gravity

The discussion of connectedness in section 2.8 had a very interesting aspect mentioned, which was that of choosing times around the Page time to canonically purify. In figure 2, we choose between two times, either pre- or post-Page time to purify.

Depending upon this, one also makes sense of entanglement wedges of the radiation and the CFT to the boundary. Together, the picture to discuss would be of purification in evaporating black holes, and the involvement of QES and JT gravity as done by Engelhardt and Folkestad.

Our interest in Part Two is in working in JT gravity, which coupled to a CFT takes the action

$$S_{JT} = \frac{1}{16\pi G_N}\left[\phi\left(\int_M R + 2\int_{\partial M} K\right) + \int_M \phi(R+2) + 2\int_{\partial M}\phi(K-1)\right] + S_{CFT}\,. \tag{3.1}$$

3.2 Page-time, Hayden-Preskill and QES

This discussion would be more at length in Part Two, since I feel that this particular aspect of QES and entanglement wedges has several correlations to the above discussed aspects of black hole information paradox. So, for now, I will introduce the core elements of Geoff Pennington's work on this.

As seen in section 3.1, one can identify a (minimal) QES in the case of an evaporating black hole to understand its dynamics. Our interest now is to see what happens when we consider different times around the Page time (as before), but in the sense of the entanglement wedge corresponding to the QES given some absorbing boundary conditions. Given such a minimal QES, from the Replica trick, we can see that the BH entropy would be the (extremized) generalized entropy of $\mathcal{X}_{QES}$:

$$S[BH,\ \mathcal{X}_{QES}] = \text{ext}\left[\frac{\text{Area of } \mathcal{X}_{QES}}{4G_N} + S_{bulk}(\mathcal{X}_{QES})\right] . \tag{3.2}$$

We will now make use of the *Hayden-Preskil* motivation. Consider the post-Page time case. where we throw in a diary into the evaporating black hole. Initially, the worldline of the diary is in the entanglement wedge of the CFT $\mathcal{E}_W(\mathcal{X}_{QES})$. In this case, there is no information about the diary from the Hawking radiation escaping the black hole. However, the RT surface moves as the black hole further evaporates. Due to this, eventually, after scrambling-time, the worldline lies in the entanglement wedge of the radiation, due to which we see that the state of the diary escapes in the Hawking radiation.

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